1

Functions

Chapter Preview Mathematics is a language with an alphabet, a vocabulary, and many rules. Before beginning your calculus journey, you should be familiar with the elements of this language. Among these elements are algebra skills; the notation and terminology for various sets of real numbers; and the descriptions of lines, circles, and other basic sets in the coordinate plane. A review of this material is found in Appendix A. This chapter begins with the fundamental concept of a function and then presents the entire cast of functions needed for calculus: polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and the trigonometric functions, along with their inverses. Before you begin studying calculus, it is important that you master the ideas in this chapter.

1.1 Review of Functions

Everywhere around us we see relationships among quantities, or **variables**. For example, the consumer price index changes in time and the temperature of the ocean varies with latitude. These relationships can often be expressed by mathematical objects called *functions*. Calculus is the study of functions, and because we use functions to describe the world around us, calculus is a universal language for human inquiry.

DEFINITION Function

A **function** f is a rule that assigns to each value x in a set D a *unique* value denoted $f(x)$. The set D is the **domain** of the function. The **range** is the set of all values of $f(x)$ produced as x varies over the entire domain (Figure 1.1).

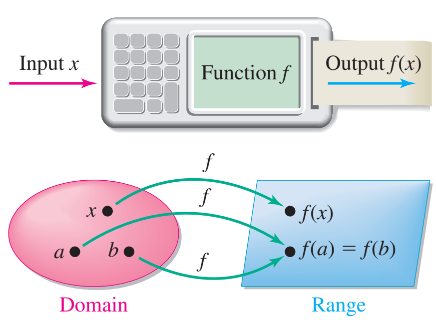


Figure 1.1

The **independent variable** is the variable associated with the domain; the **dependent variable** belongs to the range. The **graph** of a function f is the set of all points (x, y) in the xy-plane that satisfy the equation $y = f(x)$. The **argument** of a function is the expression on which the function works. For example, x is the argument when we write $f(x)$. Similarly, 2 is the argument in $f(2)$ and $x^2 + 4$ is the argument in $f(x^2 + 4)$.

QUICK CHECK 1 If $f(x) = x^2 - 2x$, find $f(-1)$, $f(x^2)$, $f(t)$, and $f(p - 1)$. ◀

The requirement that a function assigns a *unique* value of the dependent variable to each value in the domain is expressed in the vertical line test (Figure 1.2a). For example, the outside temperature as it varies over the course of a day is a function of time (Figure 1.2b).

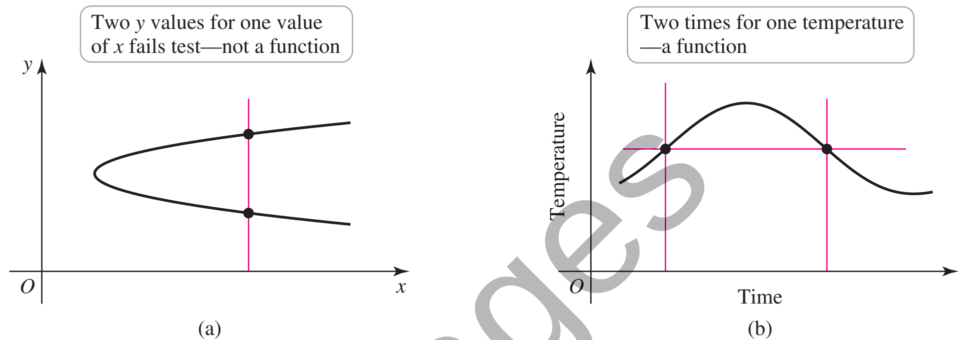


Figure 1.2

> If the domain is not specified, we take it to be the set of all values of x for which f is defined. We will see shortly that the domain and range of a function may be restricted by the context of the problem.

> A set of points or a graph that does *not* correspond to a function represents a **relation** between the variables. All functions are relations, but not all relations are functions.

Vertical Line Test

A graph represents a function if and only if it passes the **vertical line test**: Every vertical line intersects the graph at most once. A graph that fails this test does not represent a function.

EXAMPLE 1 Identifying functions State whether each graph in Figure 1.3 represents a function.

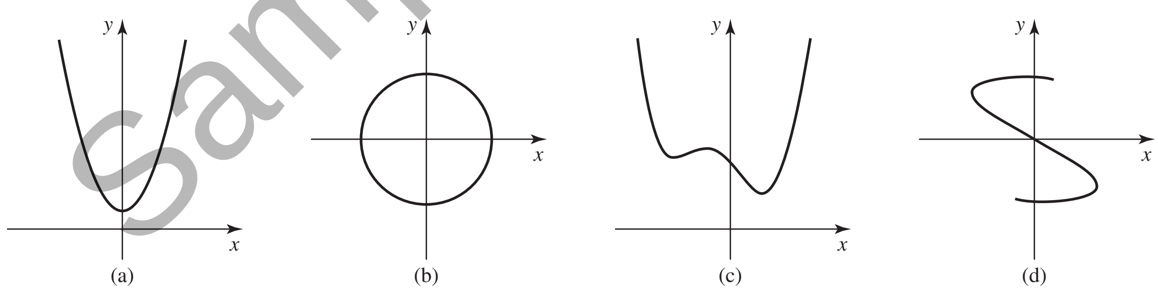


Figure 1.3

SOLUTION The vertical line test indicates that only graphs (a) and (c) represent functions. In graphs (b) and (d), there are vertical lines that intersect the graph more than once. Equivalently, there are values of x that correspond to more than one value of y. Therefore, graphs (b) and (d) do not pass the vertical line test and do not represent functions.

Related Exercises 11–12 ◀

> A window of $[a, b] \times [c, d]$ means $a \le x \le b$ and $c \le y \le d$.

EXAMPLE 2 Domain and range Graph each function with a graphing utility using the given window. Then state the domain and range of the function.

a. $y = f(x) = x^2 + 1$; $[-3, 3] \times [-1, 5]$

b. $z = g(t) = \sqrt{4 - t^2}$; $[-3, 3] \times [-1, 3]$

c. $w = h(u) = \dfrac{1}{u - 1}$; $[-3, 5] \times [-4, 4]$

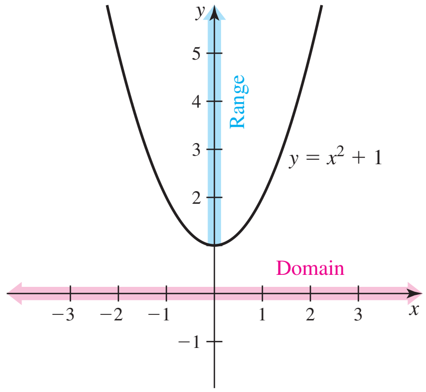


Figure 1.4

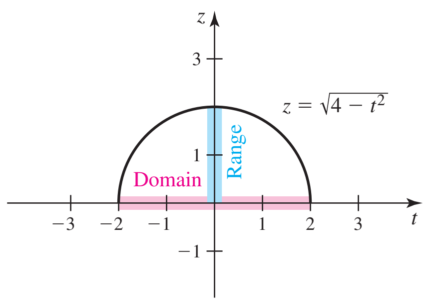


Figure 1.5

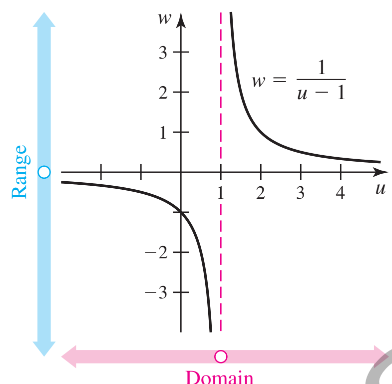


Figure 1.6

➤ The dashed vertical line $u = 1$ in Figure 1.6 indicates that the graph of $w = h(u)$ approaches a *vertical asymptote* as u approaches 1 and that w becomes large in magnitude for u near 1.

SOLUTION

a. Figure 1.4 shows the graph of $f(x) = x^2 + 1$. Because f is defined for all values of x, its domain is the set of all real numbers, written $(-\infty, \infty)$ or $\mathbb{R}$. Because $x^2 \geq 0$ for all x, it follows that $x^2 + 1 \geq 1$ and the range of f is $[1, \infty)$.

b. When n is even, functions involving nth roots are defined provided the quantity under the root is nonnegative (additional restrictions may also apply). In this case, the function g is defined provided $4 - t^2 \geq 0$, which means $t^2 \leq 4$, or $-2 \leq t \leq 2$. Therefore, the domain of g is $[-2, 2]$. By the definition of the square root, the range consists only of nonnegative numbers. When $t = 0$, z reaches its maximum value of $g(0) = \sqrt{4} = 2$, and when $t = \pm 2$, z attains its minimum value of $g(\pm 2) = 0$. Therefore, the range of g is $[0, 2]$ (Figure 1.5).

c. The function h is undefined at $u = 1$, so its domain is $\{u: u \neq 1\}$, and the graph does not have a point corresponding to $u = 1$. We see that w takes on all values except 0; therefore, the range is $\{w: w \neq 0\}$. A graphing utility does *not* represent this function accurately if it shows the vertical line $u = 1$ as part of the graph (Figure 1.6).

Related Exercises 13–20 ◄

EXAMPLE 3 **Domain and range in context** At time $t = 0$, a stone is thrown vertically upward from the ground at a speed of 30 m/s. Its height above the ground in meters (neglecting air resistance) is approximated by the function $h = f(t) = 30t - 5t^2$, where t is measured in seconds. Find the domain and range of f in the context of this particular problem.

SOLUTION Although f is defined for all values of t, the only relevant times are between the time the stone is thrown $(t = 0)$ and the time it strikes the ground, when $h = f(t) = 0$. Solving the equation $h = 30t - 5t^2 = 0$, we find that

$$
\begin{aligned}
30t - 5t^2 &= 0 \\
5t(6 - t) &= 0 && \text{Factor.} \\
5t = 0 \quad &\text{or} \quad 6 - t = 0 && \text{Set each factor equal to 0.} \\
t = 0 \quad &\text{or} \quad t = 6. && \text{Solve.}
\end{aligned}
$$

Therefore, the stone leaves the ground at $t = 0$ and returns to the ground at $t = 6$. An appropriate domain that fits the context of this problem is $\{t: 0 \leq t \leq 6\}$. The range consists of all values of $h = 30t - 5t^2$ as t varies over $[0, 6]$. The largest value of h occurs when the stone reaches its highest point at $t = 3$ (halfway through its flight), which is $h = f(3) = 45$. Therefore, the range is $[0, 45]$. These observations are confirmed by the graph of the height function (Figure 1.7). Note that this graph is *not* the trajectory of the stone; the stone moves vertically.

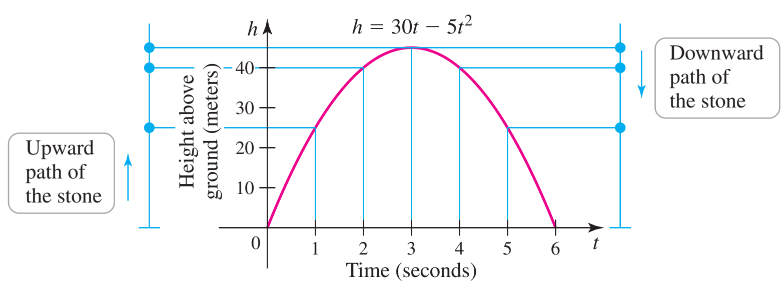


Figure 1.7

Related Exercises 21–24 ◄

QUICK CHECK 2 State the domain and range of $f(x) = (x^2 + 1)^{-1}$. ◄

Composite Functions

Functions may be combined using sums $(f + g)$, differences $(f - g)$, products (fg), or quotients (f/g). The process called *composition* also produces new functions.

➤ In the composition $y = f(g(x))$, f is the *outer function* and g is the *inner function*.

DEFINITION Composite Functions

Given two functions f and g, the composite function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. It is evaluated in two steps: $y = f(u)$, where $u = g(x)$. The domain of $f \circ g$ consists of all x in the domain of g such that $u = g(x)$ is in the domain of f (Figure 1.8).

➤ You have now seen three different notations for intervals on the real number line, all of which will be used throughout the book:

- $[-2, 3)$ is an example of interval notation,
- $-2 \le x < 3$ is inequality notation, and
- $\{x: -2 \le x < 3\}$ is set notation.

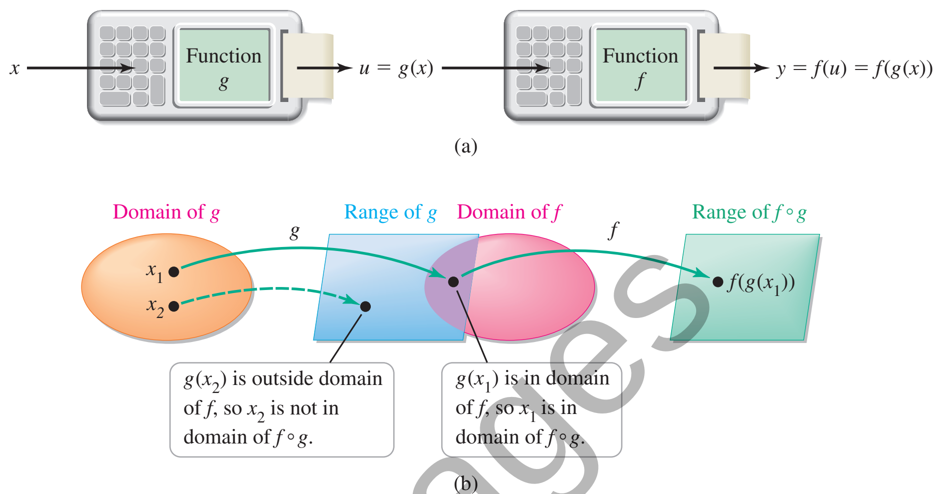


Figure 1.8

EXAMPLE 4 Composite functions and notation Let $f(x) = 3x^2 - x$ and $g(x) = 1/x$. Simplify the following expressions.

a. $f(5p + 1)$ **b.** $g(1/x)$ **c.** $f(g(x))$ **d.** $g(f(x))$

SOLUTION In each case, the functions work on their arguments.

a. The argument of f is $5p + 1$, so

$$f(5p + 1) = 3(5p + 1)^2 - (5p + 1) = 75p^2 + 25p + 2.$$

b. Because g requires taking the reciprocal of the argument, we take the reciprocal of $1/x$ and find that $g(1/x) = 1/(1/x) = x$.

c. The argument of f is $g(x)$, so

➤ Examples 4c and 4d demonstrate that, in general,

$$f(g(x)) \ne g(f(x)).$$

$$f(g(x)) = f\left(\frac{1}{x}\right) = 3\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right) = \frac{3}{x^2} - \frac{1}{x} = \frac{3 - x}{x^2}.$$

d. The argument of g is $f(x)$, so

$$g(f(x)) = g(3x^2 - x) = \frac{1}{3x^2 - x}.$$

Related Exercises 25–36 ◀

EXAMPLE 5 Working with composite functions Identify possible choices for the inner and outer functions in the following composite functions. Give the domain of the composite function.

a. $h(x) = \sqrt{9x - x^2}$ **b.** $h(x) = \dfrac{2}{(x^2 - 1)^3}$

SOLUTION

a. An obvious outer function is $f(x) = \sqrt{x}$, which works on the inner function $g(x) = 9x - x^2$. Therefore, h can be expressed as $h = f \circ g$ or $h(x) = f(g(x))$. The domain of $f \circ g$ consists of all values of x such that $9x - x^2 \ge 0$. Solving this inequality gives $\{x: 0 \le x \le 9\}$ as the domain of $f \circ g$.

➤ Techniques for solving inequalities are discussed in Appendix A.

b. A good choice for an outer function is $f(x) = 2/x^3 = 2x^{-3}$, which works on the inner function $g(x) = x^2 - 1$. Therefore, h can be expressed as $h = f \circ g$ or $h(x) = f(g(x))$. The domain of $f \circ g$ consists of all values of $g(x)$ such that $g(x) \neq 0$, which is $\{x: x \neq \pm 1\}$.

Related Exercises 37–40 ◄

EXAMPLE 6 **More composite functions** Given $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - x - 6$, find (a) $g \circ f$ and (b) $g \circ g$, and their domains.

SOLUTION

a. We have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt[3]{x}) = \underbrace{(\sqrt[3]{x})}_{f(x)}{}^2 - \underbrace{\sqrt[3]{x}}_{f(x)} - 6 = x^{2/3} - x^{1/3} - 6.$$

Because the domains of f and g are $(-\infty, \infty)$, the domain of $f \circ g$ is also $(-\infty, \infty)$.

b. In this case, we have the composition of two polynomials:

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(x^2 - x - 6) \\ &= \underbrace{(x^2 - x - 6)}_{g(x)}{}^2 - \underbrace{(x^2 - x - 6)}_{g(x)} - 6 \\ &= x^4 - 2x^3 - 12x^2 + 13x + 36.\end{aligned}$$

The domain of the composition of two polynomials is $(-\infty, \infty)$.

Related Exercises 41–54 ◄

QUICK CHECK 3 If $f(x) = x^2 + 1$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. ◄

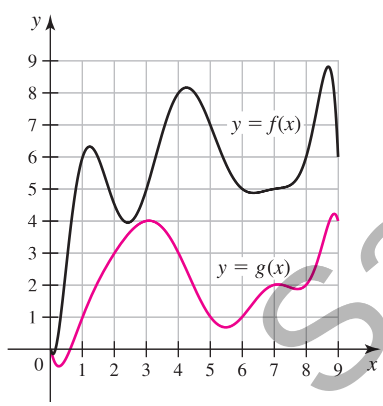


Figure 1.9

EXAMPLE 7 **Using graphs to evaluate composite functions** Use the graphs of f and g in Figure 1.9 to find the following values.

a. $f(g(3))$ **b.** $g(f(3))$ **c.** $f(f(4))$ **d.** $f(g(f(8)))$

SOLUTION

a. The graphs indicate that $g(3) = 4$ and $f(4) = 8$, so $f(g(3)) = f(4) = 8$.

b. We see that $g(f(3)) = g(5) = 1$. Observe that $f(g(3)) \neq g(f(3))$.

c. In this case, $f(\underbrace{f(4)}_{8}) = f(8) = 6$.

d. Starting on the inside,

$$f(g(\underbrace{f(8)}_{6})) = f(\underbrace{g(6)}_{1}) = f(1) = 6.$$

Related Exercises 55–56 ◄

EXAMPLE 8 **Using a table to evaluate composite functions** Use the function values in the table to evaluate the following composite functions.

a. $(f \circ g)(0)$ **b.** $g(f(-1))$ **c.** $f(g(g(-1)))$

x	-2	-1	0	1	2
$f(x)$	0	1	3	4	2
$g(x)$	-1	0	-2	-3	-4

SOLUTION

a. Using the table, we see that $g(0) = -2$ and $f(-2) = 0$. Therefore, $(f \circ g)(0) = 0$.

b. Because $f(-1) = 1$ and $g(1) = -3$, it follows that $g(f(-1)) = -3$.

c. Starting with the inner function,

$$f(g(\underbrace{g(-1)}_{0})) = f(\underbrace{g(0)}_{-2}) = f(-2) = 0.$$

Related Exercises 55–56 ◀

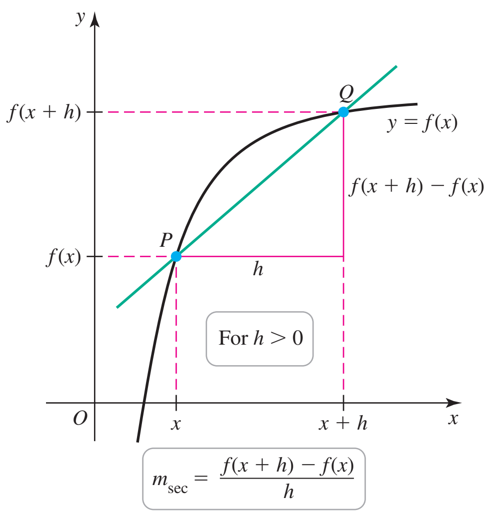


Figure 1.10

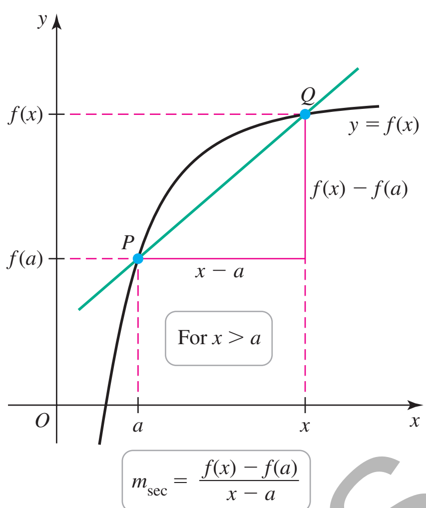


Figure 1.11

➤ Treat $f(x + h)$ like the composition $f(g(x))$, where $x + h$ plays the role of $g(x)$. It may help to establish a pattern in your mind before evaluating $f(x + h)$. For instance, using the function in Example 9a, we have

$$\begin{aligned} f(x) &= 3x^2 - x; \\ f(12) &= 3 \cdot 12^2 - 12; \\ f(b) &= 3b^2 - b; \\ f(\text{math}) &= 3 \cdot \text{math}^2 - \text{math}; \end{aligned}$$

therefore,

$$f(x + h) = 3(x + h)^2 - (x + h).$$

Secant Lines and the Difference Quotient

As you will see shortly, slopes of lines and curves play a fundamental role in calculus. Figure 1.10 shows two points $P(x, f(x))$ and $Q(x + h, f(x + h))$ on the graph of $y = f(x)$ in the case that $h > 0$. A line through any two points on a curve is called a **secant line**; its importance in the study of calculus is explained in Chapters 2 and 3. For now, we focus on the slope of the secant line through P and Q, which is denoted m_{sec} and is given by

$$m_{\text{sec}} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.$$

The slope formula $\dfrac{f(x + h) - f(x)}{h}$ is also known as a **difference quotient**, and it can be expressed in several ways depending on how the coordinates of P and Q are labeled. For example, given the coordinates $P(a, f(a))$ and $Q(x, f(x))$ (Figure 1.11), the difference quotient is

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We interpret the slope of the secant line in this form as the **average rate of change** of f over the interval $[a, x]$.

EXAMPLE 9 **Working with difference quotients**

a. Simplify the difference quotient $\dfrac{f(x + h) - f(x)}{h}$, for $f(x) = 3x^2 - x$.

b. Simplify the difference quotient $\dfrac{f(x) - f(a)}{x - a}$, for $f(x) = x^3$.

SOLUTION

a. First note that $f(x + h) = 3(x + h)^2 - (x + h)$. We substitute this expression into the difference quotient and simplify:

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{\overbrace{3(x + h)^2 - (x + h)}^{f(x + h)} - \overbrace{(3x^2 - x)}^{f(x)}}{h} \\ &= \frac{3(x^2 + 2xh + h^2) - (x + h) - (3x^2 - x)}{h} && \text{Expand } (x + h)^2. \\ &= \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h} && \text{Distribute.} \\ &= \frac{6xh + 3h^2 - h}{h} && \text{Simplify.} \\ &= \frac{h(6x + 3h - 1)}{h} = 6x + 3h - 1. && \text{Factor and simplify.} \end{aligned}$$

➤ Some useful factoring formulas:

1. Difference of perfect squares:
$$x^2 - y^2 = (x - y)(x + y).$$
2. Sum of perfect squares: $x^2 + y^2$ does not factor over the real numbers.
3. Difference of perfect cubes:
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$
4. Sum of perfect cubes:
$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

b. The factoring formula for the difference of perfect cubes is needed:

$$\begin{aligned}\frac{f(x) - f(a)}{x - a} &= \frac{x^3 - a^3}{x - a} \\ &= \frac{(x - a)(x^2 + ax + a^2)}{x - a} && \text{Factoring formula} \\ &= x^2 + ax + a^2. && \text{Simplify.}\end{aligned}$$

Related Exercises 57–66 ◄

EXAMPLE 10 Interpreting the slope of the secant line Sound intensity I, measured in watts per square meter $(\mathrm{W/m^2})$, at a point r meters from a sound source with acoustic power P is given by $I(r) = \dfrac{P}{4\pi r^2}$.

a. Find the sound intensity at two points $r_1 = 10$ m and $r_2 = 15$ m from a sound source with power $P = 100$ W. Then find the slope of the secant line through the points $(10, I(10))$ and $(15, I(15))$ on the graph of the intensity function and interpret the result.

b. Find the slope of the secant line through any two points $(r_1, I(r_1))$ and $(r_2, I(r_2))$ on the graph of the intensity function with acoustic power P.

SOLUTION

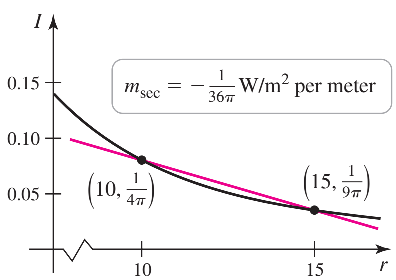


Figure 1.12

a. The sound intensity 10 m from the source is $I(10) = \dfrac{100\text{ W}}{4\pi(10\text{ m})^2} = \dfrac{1}{4\pi}\,\mathrm{W/m^2}$. At 15 m, the intensity is $I(15) = \dfrac{100\text{ W}}{4\pi(15\text{ m})^2} = \dfrac{1}{9\pi}\,\mathrm{W/m^2}$. To find the slope of the secant line (Figure 1.12), we compute the change in intensity divided by the change in distance:

$$m_{\sec} = \frac{I(15) - I(10)}{15 - 10} = \frac{\frac{1}{9\pi} - \frac{1}{4\pi}}{5} = -\frac{1}{36\pi} \approx -0.0088\ \mathrm{W/m^2}\text{ per meter.}$$

The units provide a clue to the physical meaning of the slope: It measures the average rate at which the intensity changes as one moves from 10 m to 15 m away from the sound source. In this case, because the slope of the secant line is negative, the intensity *decreases* (slowly) at an average rate of $1/(36\pi)\ \mathrm{W/m^2}$ per meter.

b.

$$\begin{aligned}m_{\sec} = \frac{I(r_2) - I(r_1)}{r_2 - r_1} &= \frac{\frac{P}{4\pi r_2^2} - \frac{P}{4\pi r_1^2}}{r_2 - r_1} && \text{Evaluate } I(r_2) \text{ and } I(r_1). \\ &= \frac{\frac{P}{4\pi}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)}{r_2 - r_1} && \text{Factor.} \\ &= \frac{P}{4\pi}\left(\frac{r_1^2 - r_2^2}{r_1^2 r_2^2}\right)\frac{1}{r_2 - r_1} && \text{Simplify.} \\ &= \frac{P}{4\pi} \cdot \frac{(r_1 - r_2)(r_1 + r_2)}{r_1^2 r_2^2} \cdot \frac{1}{-(r_1 - r_2)} && \text{Factor.} \\ &= -\frac{P(r_1 + r_2)}{4\pi r_1^2 r_2^2} && \text{Cancel and simplify.}\end{aligned}$$

This result is the average rate at which the sound intensity changes over an interval $[r_1, r_2]$. Because $r_1 > 0$ and $r_2 > 0$, we see that $m_{\sec}$ is always negative. Therefore, the sound intensity $I(r)$ decreases as r increases, for $r > 0$.

Related Exercises 67–70 ◄

Symmetry

The word *symmetry* has many meanings in mathematics. Here we consider symmetries of graphs and the relations they represent. Taking advantage of symmetry often saves time and leads to insights.

> **DEFINITION Symmetry in Graphs**
>
> A graph is **symmetric with respect to the y-axis** if whenever the point (x, y) is on the graph, the point $(-x, y)$ is also on the graph. This property means that the graph is unchanged when reflected across the y-axis (Figure 1.13a).
>
> A graph is **symmetric with respect to the x-axis** if whenever the point (x, y) is on the graph, the point $(x, -y)$ is also on the graph. This property means that the graph is unchanged when reflected across the x-axis (Figure 1.13b).
>
> A graph is **symmetric with respect to the origin** if whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph (Figure 1.13c). Symmetry about both the x- and y-axes implies symmetry about the origin, but not vice versa.

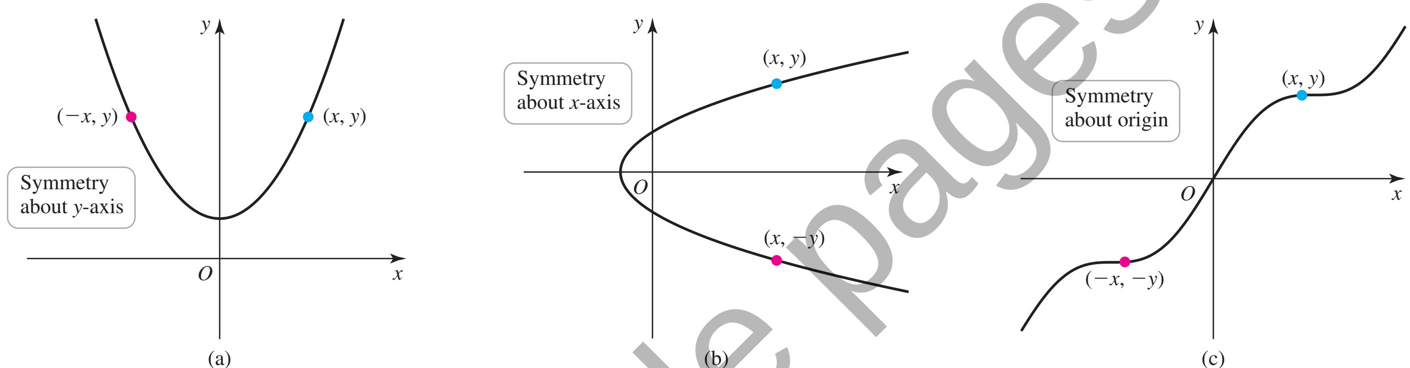


Figure 1.13

> **DEFINITION Symmetry in Functions**
>
> An **even function** f has the property that $f(-x) = f(x)$, for all x in the domain. The graph of an even function is symmetric about the y-axis.
>
> An **odd function** f has the property that $f(-x) = -f(x)$, for all x in the domain. The graph of an odd function is symmetric about the origin.

Polynomials consisting of only even powers of the variable (of the form x^{2n}, where n is a nonnegative integer) are even functions. Polynomials consisting of only odd powers of the variable (of the form x^{2n+1}, where n is a nonnegative integer) are odd functions.

QUICK CHECK 4 Explain why the graph of a nonzero function is never symmetric with respect to the x-axis. ◄

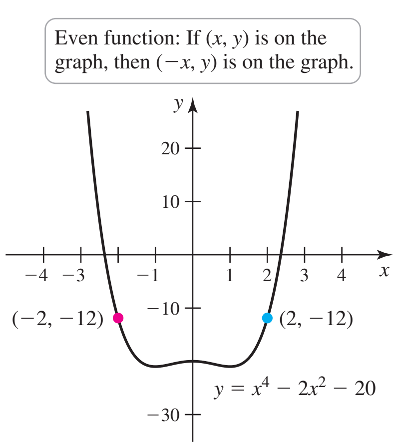


Figure 1.14

EXAMPLE 11 **Identifying symmetry in functions** Identify the symmetry, if any, in the following functions.

a. $f(x) = x^4 - 2x^2 - 20$ **b.** $g(x) = x^3 - 3x + 1$ **c.** $h(x) = \dfrac{1}{x^3 - x}$

SOLUTION

a. The function f consists of only even powers of x (where $20 = 20 \cdot 1 = 20x^0$ and x^0 is considered an even power). Therefore, f is an even function (Figure 1.14). This fact is verified by showing that $f(-x) = f(x)$:

$$f(-x) = (-x)^4 - 2(-x)^2 - 20 = x^4 - 2x^2 - 20 = f(x).$$

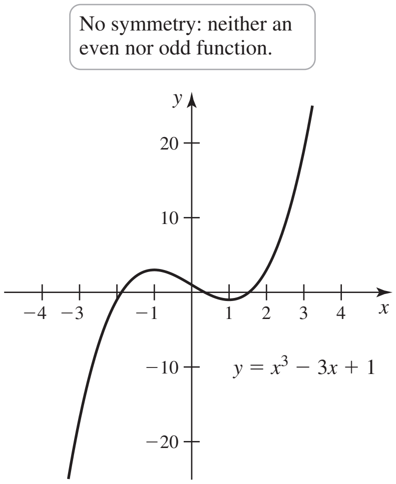


Figure 1.15

➤ The symmetry of compositions of even and odd functions is considered in Exercises 95–101.

b. The function g consists of two odd powers and one even power (again, $1 = x^0$ is an even power). Therefore, we expect that g has no symmetry about the y-axis or the origin (Figure 1.15). Note that

$$g(-x) = (-x)^3 - 3(-x) + 1 = -x^3 + 3x + 1,$$

so $g(-x)$ equals neither $g(x)$ nor $-g(x)$; therefore, g has no symmetry.

c. In this case, h is a composition of an odd function $f(x) = 1/x$ with an odd function $g(x) = x^3 - x$. Note that

$$h(-x) = \frac{1}{(-x)^3 - (-x)} = -\frac{1}{x^3 - x} = -h(x).$$

Because $h(-x) = -h(x)$, h is an odd function (Figure 1.16).

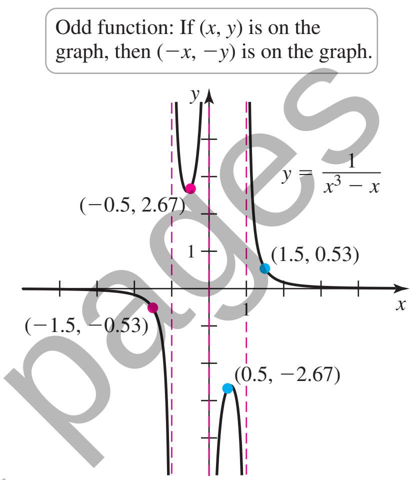


Figure 1.16

Related Exercises 71–80 ◀

SECTION 1.1 EXERCISES

Review Questions

1. Use the terms *domain, range, independent variable*, and *dependent variable* to explain how a function relates one variable to another variable.

2. Is the independent variable of a function associated with the domain or range? Is the dependent variable associated with the domain or range?

3. Explain how the vertical line test is used to detect functions.

4. If $f(x) = 1/(x^3 + 1)$, what is $f(2)$? What is $f(y^2)$?

5. Which statement about a function is true? (i) For each value of x in the domain, there corresponds one unique value of y in the range; (ii) for each value of y in the range, there corresponds one unique value of x in the domain. Explain.

6. If $f(x) = \sqrt{x}$ and $g(x) = x^3 - 2$, find the compositions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

7. Suppose f and g are even functions with $f(2) = 2$ and $g(2) = -2$. Evaluate $f(g(2))$ and $g(f(-2))$.

8. Explain how to find the domain of $f \circ g$ if you know the domain and range of f and g.

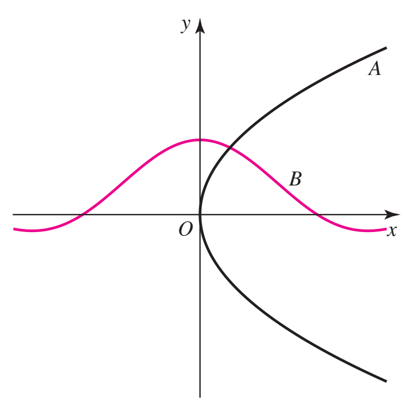
9. Sketch a graph of an even function f and state how $f(x)$ and $f(-x)$ are related.

10. Sketch a graph of an odd function f and state how $f(x)$ and $f(-x)$ are related.

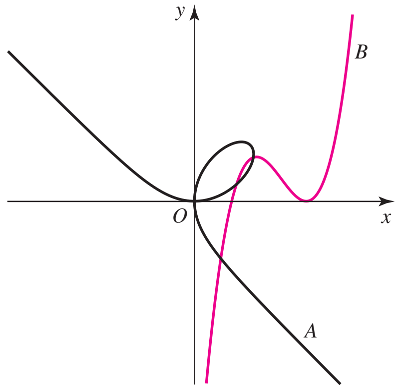
Basic Skills

11–12. Vertical line test *Decide whether graphs A, B, or both represent functions.*

11.



12.



13–20. Domain and range *Graph each function with a graphing utility using the given window. Then state the domain and range of the function.*

13. $f(x) = 3x^4 - 10;\quad [-2, 2] \times [-10, 15]$

14. $g(y) = \dfrac{y + 1}{(y + 2)(y - 3)};\quad [-4, 6] \times [-3, 3]$

15. $f(x) = \sqrt{4 - x^2};\quad [-4, 4] \times [-4, 4]$

16. $F(w) = \sqrt[4]{2 - w};\quad [-3, 2] \times [0, 2]$

17. $h(u) = \sqrt[3]{u - 1};\quad [-7, 9] \times [-2, 2]$

18. $g(x) = (x^2 - 4)\sqrt{x + 5};\quad [-5, 5] \times [-10, 50]$

19. $f(x) = (9 - x^2)^{3/2};\quad [-4, 4] \times [0, 30]$

20. $g(t) = \dfrac{1}{1 + t^2};\quad [-7, 7] \times [0, 1.5]$

21–24. Domain in context *Determine an appropriate domain of each function. Identify the independent and dependent variables.*

21. A stone is thrown vertically upward from the ground at a speed of 40 m/s at time $t = 0$. Its distance d (in meters) above the ground (neglecting air resistance) is approximated by the function $f(t) = 40t - 5t^2$.

22. A stone is dropped off a bridge from a height of 20 m above a river. If t represents the elapsed time (in seconds) after the stone is released, then its distance d (in meters) above the river is approximated by the function $f(t) = 20 - 5t^2$.

23. A cylindrical water tower with a radius of 10 m and a height of 50 m is filled to a height of h. The volume V of water (in cubic meters) is given by the function $g(h) = 100\pi h$.

24. The volume V of a balloon of radius r (in meters) filled with helium is given by the function $f(r) = \frac{4}{3}\pi r^3$. Assume the balloon can hold up to 1 m^3 of helium.

25–36. Composite functions and notation *Let* $f(x) = x^2 - 4$, $g(x) = x^3$, *and* $F(x) = 1/(x - 3)$. *Simplify or evaluate the following expressions.*

25. $f(10)$ **26.** $f(p^2)$ **27.** $g(1/z)$

28. $F(y^4)$ **29.** $F(g(y))$ **30.** $f(g(w))$

31. $g(f(u))$ **32.** $\dfrac{f(2 + h) - f(2)}{h}$ **33.** $F(F(x))$

34. $g(F(f(x)))$ **35.** $f\left(\sqrt{x + 4}\right)$ **36.** $F\left(\dfrac{3x + 1}{x}\right)$

37–40. Working with composite functions *Find possible choices for the outer and inner functions f and g such that the given function h equals* $f \circ g$. *Give the domain of h.*

37. $h(x) = (x^3 - 5)^{10}$

38. $h(x) = \dfrac{2}{(x^6 + x^2 + 1)^2}$

39. $h(x) = \sqrt{x^4 + 2}$

40. $h(x) = \dfrac{1}{\sqrt{x^3 - 1}}$

41–48. More composite functions *Let* $f(x) = |x|$, $g(x) = x^2 - 4$, $F(x) = \sqrt{x}$, *and* $G(x) = 1/(x - 2)$. *Determine the following composite functions and give their domains.*

41. $f \circ g$ **42.** $g \circ f$ **43.** $f \circ G$

44. $f \circ g \circ G$ **45.** $G \circ g \circ f$ **46.** $F \circ g \circ g$

47. $g \circ g$ **48.** $G \circ G$

49–54. Missing piece *Let* $g(x) = x^2 + 3$. *Find a function f that produces the given composition.*

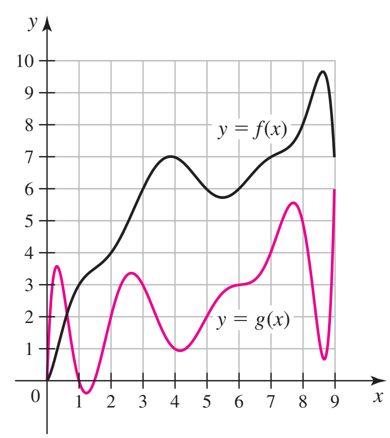
49. $(f \circ g)(x) = x^2$ **50.** $(f \circ g)(x) = \dfrac{1}{x^2 + 3}$

51. $(f \circ g)(x) = x^4 + 6x^2 + 9$ **52.** $(f \circ g)(x) = x^4 + 6x^2 + 20$

53. $(g \circ f)(x) = x^4 + 3$ **54.** $(g \circ f)(x) = x^{2/3} + 3$

55. Composite functions from graphs Use the graphs of f and g in the figure to determine the following function values.

a. $(f \circ g)(2)$ **b.** $g(f(2))$ **c.** $f(g(4))$
d. $g(f(5))$ **e.** $f(f(8))$ **f.** $g(f(g(5)))$



56. Composite functions from tables Use the table to evaluate the given compositions.

x	-1	0	1	2	3	4
$f(x)$	3	1	0	-1	-3	-1
$g(x)$	-1	0	2	3	4	5
$h(x)$	0	-1	0	3	0	4

a. $h(g(0))$ **b.** $g(f(4))$ **c.** $h(h(0))$
d. $g(h(f(4)))$ **e.** $f(f(f(1)))$ **f.** $h(h(h(0)))$
g. $f(h(g(2)))$ **h.** $g(f(h(4)))$ **i.** $g(g(g(1)))$
j. $f(f(h(3)))$

57–61. Working with difference quotients *Simplify the difference quotient* $\dfrac{f(x+h)-f(x)}{h}$ *for the following functions.*

57. $f(x) = x^2$

58. $f(x) = 4x - 3$

59. $f(x) = 2/x$

60. $f(x) = 2x^2 - 3x + 1$

61. $f(x) = \dfrac{x}{x+1}$

62–66. Working with difference quotients *Simplify the difference quotient* $\dfrac{f(x)-f(a)}{x-a}$ *for the following functions.*

62. $f(x) = x^4$

63. $f(x) = x^3 - 2x$

64. $f(x) = 4 - 4x - x^2$

65. $f(x) = -\dfrac{4}{x^2}$

66. $f(x) = \dfrac{1}{x} - x^2$

67–70. Interpreting the slope of secant lines *In each exercise, a function and an interval of its independent variable are given. The endpoints of the interval are associated with the points P and Q on the graph of the function.*

a. *Sketch a graph of the function and the secant line through P and Q.*

b. *Find the slope of the secant line in part (a), and interpret your answer in terms of an average rate of change over the interval. Include units in your answer.*

67. After t seconds, an object dropped from rest falls a distance $d = 16t^2$, where d is measured in feet and $2 \le t \le 5$.

68. After t seconds, the second hand on a clock moves through an angle $D = 6t$, where D is measured in degrees and $5 \le t \le 20$.

69. The volume V of an ideal gas in cubic centimeters is given by $V = 2/p$, where p is the pressure in atmospheres and $0.5 \le p \le 2$.

70. The speed of a car prior to hard braking can be estimated by the length of the skid mark. One model claims that the speed S in mi/hr is $S = \sqrt{30\ell}$, where ℓ is the length of the skid mark in feet and $50 \le \ell \le 150$.

T 71–78. Symmetry *Determine whether the graphs of the following equations and functions are symmetric about the x-axis, the y-axis, or the origin. Check your work by graphing.*

71. $f(x) = x^4 + 5x^2 - 12$

72. $f(x) = 3x^5 + 2x^3 - x$

73. $f(x) = x^5 - x^3 - 2$

74. $f(x) = 2|x|$

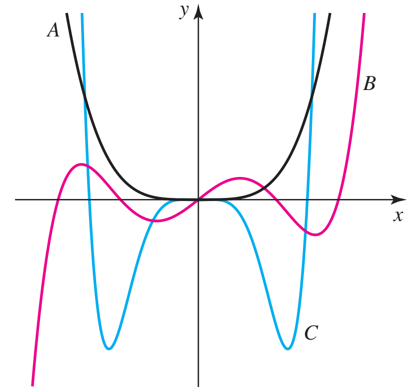
75. $x^{2/3} + y^{2/3} = 1$

76. $x^3 - y^5 = 0$

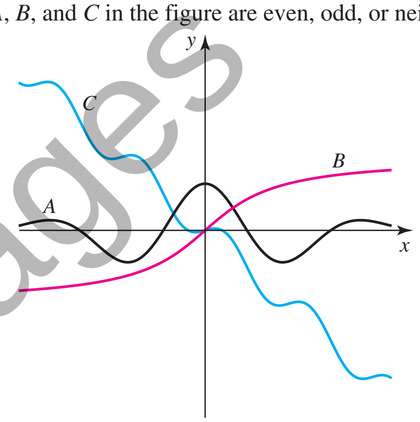
77. $f(x) = x|x|$

78. $|x| + |y| = 1$

79. Symmetry in graphs State whether the functions represented by graphs A, B, and C in the figure are even, odd, or neither.



80. Symmetry in graphs State whether the functions represented by graphs A, B, and C in the figure are even, odd, or neither.



Further Explorations

81. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The range of $f(x) = 2x - 38$ is all real numbers.

b. The relation $y = x^6 + 1$ is *not* a function because $y = 2$ for both $x = -1$ and $x = 1$.

c. If $f(x) = x^{-1}$, then $f(1/x) = 1/f(x)$.

d. In general, $f(f(x)) = (f(x))^2$.

e. In general, $f(g(x)) = g(f(x))$.

f. By definition, $f(g(x)) = (f \circ g)(x)$.

g. If $f(x)$ is an even function, then $c\,f(ax)$ is an even function, where a and c are nonzero real numbers.

h. If $f(x)$ is an odd function, then $f(x) + d$ is an odd function, where d is a nonzero real number.

i. If f is both even *and* odd, then $f(x) = 0$ for all x.

82. Range of power functions Using words and figures, explain why the range of $f(x) = x^n$, where n is a positive odd integer, is all real numbers. Explain why the range of $g(x) = x^n$, where n is a positive even integer, is all nonnegative real numbers.

T 83. Absolute value graph Use the definition of absolute value to graph the equation $|x| - |y| = 1$. Use a graphing utility to check your work.

84. Even and odd at the origin

a. If $f(0)$ is defined and f is an even function, is it necessarily true that $f(0) = 0$? Explain.

b. If $f(0)$ is defined and f is an odd function, is it necessarily true that $f(0) = 0$? Explain.

85–88. Polynomial calculations *Find a polynomial f that satisfies the following properties. (Hint: Determine the degree of f; then substitute a polynomial of that degree and solve for its coefficients.)*

85. $f(f(x)) = 9x - 8$ **86.** $(f(x))^2 = 9x^2 - 12x + 4$

87. $f(f(x)) = x^4 - 12x^2 + 30$ **88.** $(f(x))^2 = x^4 - 12x^2 + 36$

89–92. Difference quotients *Simplify the difference quotients* $\dfrac{f(x+h) - f(x)}{h}$ *and* $\dfrac{f(x) - f(a)}{x - a}$ *by rationalizing the numerator.*

89. $f(x) = \sqrt{x}$ **90.** $f(x) = \sqrt{1 - 2x}$

91. $f(x) = -\dfrac{3}{\sqrt{x}}$ **92.** $f(x) = \sqrt{x^2 + 1}$

Applications

93. Launching a rocket A small rocket is launched vertically upward from the edge of a cliff 80 ft off the ground at a speed of 96 ft/s. Its height (in feet) above the ground is given by $h(t) = -16t^2 + 96t + 80$, where t represents time measured in seconds.

a. Assuming the rocket is launched at $t = 0$, what is an appropriate domain for h?
b. Graph h and determine the time at which the rocket reaches its highest point. What is the height at that time?

94. Draining a tank (Torricelli's law) A cylindrical tank with a cross-sectional area of 100 cm^2 is filled to a depth of 100 cm with water. At $t = 0$, a drain in the bottom of the tank with an area of 10 cm^2 is opened, allowing water to flow out of the tank. The depth of water in the tank at time $t \geq 0$ is $d(t) = (10 - 2.2t)^2$.

a. Check that $d(0) = 100$, as specified.
b. At what time is the tank empty?
c. What is an appropriate domain for d?

Additional Exercises

95–101. Combining even and odd functions *Let E be an even function and O be an odd function. Determine the symmetry, if any, of the following functions.*

95. $E + O$ **96.** $E \cdot O$ **97.** E/O **98.** $E \circ O$

99. $E \circ E$ **100.** $O \circ O$ **101.** $O \circ E$

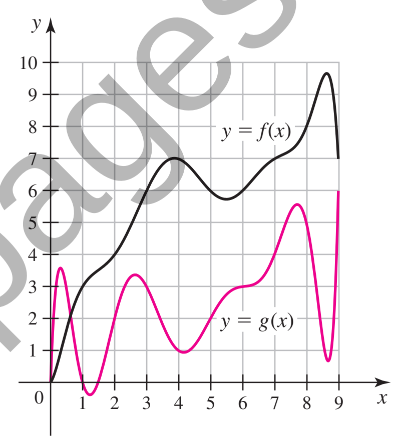
102. Composition of even and odd functions from tables Assume f is an even function and g is an odd function. Use the (incomplete) table to evaluate the given compositions.

x	1	2	3	4
$f(x)$	2	-1	3	-4
$g(x)$	-3	-1	-4	-2

a. $f(g(-1))$ **b.** $g(f(-4))$ **c.** $f(g(-3))$
d. $f(g(-2))$ **e.** $g(g(-1))$ **f.** $f(g(0) - 1)$
g. $f(g(g(-2)))$ **h.** $g(f(f(-4)))$ **i.** $g(g(g(-1)))$

103. Composition of even and odd functions from graphs Assume f is an even function and g is an odd function. Use the (incomplete) graphs of f and g in the figure to determine the following function values.

a. $f(g(-2))$ **b.** $g(f(-2))$ **c.** $f(g(-4))$
d. $g(f(5) - 8)$ **e.** $g(g(-7))$ **f.** $f(1 - f(8))$



QUICK CHECK ANSWERS

1. $3, x^4 - 2x^2, t^2 - 2t, p^2 - 4p + 3$ **2.** Domain is all real numbers; range is $\{y: 0 < y \leq 1\}$.
3. $(f \circ g)(x) = x^4 + 1$ and $(g \circ f)(x) = (x^2 + 1)^2$
4. If the graph were symmetric with respect to the x-axis, it would not pass the vertical line test. ◄

1.2 Representing Functions

We consider four approaches to defining and representing functions: formulas, graphs, tables, and words.

Using Formulas

The following list is a brief catalog of the families of functions that are introduced in this chapter and studied systematically throughout this book; they are all defined by *formulas*.

➤ One version of the Fundamental Theorem of Algebra states that a nonzero polynomial of degree n has exactly n (possibly complex) roots, counting each root up to its multiplicity.

1. Polynomials are functions of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$ and the nonnegative integer n is the **degree** of the polynomial. The domain of any polynomial is the set of all real numbers. An nth-degree polynomial can have as many as n real **zeros** or **roots**—values of x at which $p(x) = 0$; the zeros are points at which the graph of p intersects the x-axis.

2. **Rational functions** are ratios of the form $f(x) = p(x)/q(x)$, where p and q are polynomials. Because division by zero is prohibited, the domain of a rational function is the set of all real numbers except those for which the denominator is zero.
3. **Algebraic functions** are constructed using the operations of algebra: addition, subtraction, multiplication, division, and roots. Examples of algebraic functions are $f(x) = \sqrt{2x^3 + 4}$ and $g(x) = x^{1/4}(x^3 + 2)$. In general, if an even root (square root, fourth root, and so forth) appears, then the domain does not contain points at which the quantity under the root is negative (and perhaps other points).

➤ Exponential and logarithmic functions are introduced in Section 1.3.

4. **Exponential functions** have the form $f(x) = b^x$, where the base $b \neq 1$ is a positive real number. Closely associated with exponential functions are **logarithmic functions** of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$. Exponential functions have a domain consisting of all real numbers. Logarithmic functions are defined for positive real numbers.

   The **natural exponential function** is $f(x) = e^x$, with base $b = e$, where $e \approx 2.71828\ldots$ is one of the fundamental constants of mathematics. Associated with the natural exponential function is the **natural logarithm function** $f(x) = \ln x$, which also has the base $b = e$.

➤ Trigonometric functions and their inverses are introduced in Section 1.4.

5. The **trigonometric functions** are $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$; they are fundamental to mathematics and many areas of application. Also important are their relatives, the **inverse trigonometric functions**.
6. Trigonometric, exponential, and logarithmic functions are a few examples of a large family called **transcendental functions**. Figure 1.17 shows the organization of these functions, which are explored in detail in upcoming chapters.

QUICK CHECK 1 Are all polynomials rational functions? Are all algebraic functions polynomials? ◄

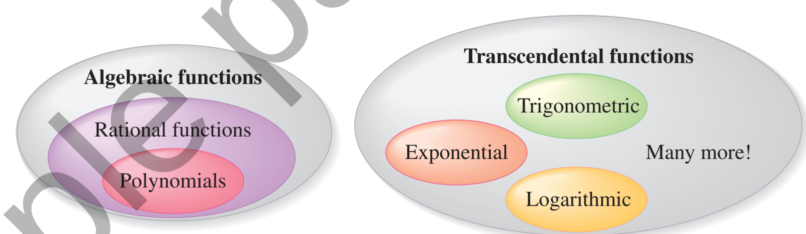


Figure 1.17

Using Graphs

Although formulas are the most compact way to represent many functions, graphs often provide the most illuminating representations. Two of countless examples of functions and their graphs are shown in Figure 1.18. Much of this book is devoted to creating and analyzing graphs of functions.

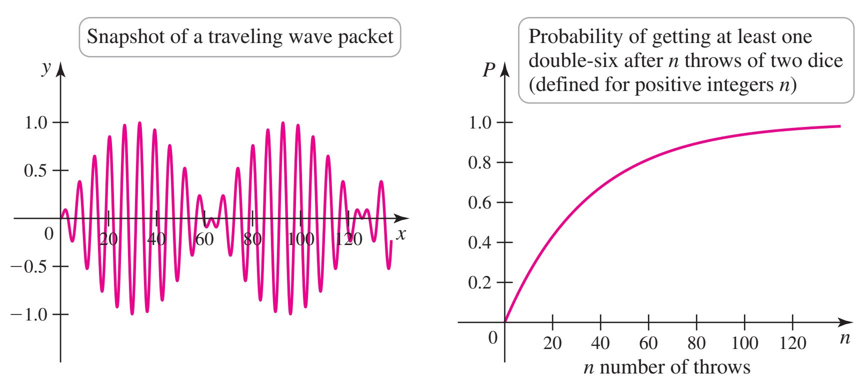


Figure 1.18

There are two approaches to graphing functions.

- Graphing calculators, tablets, and software are easy to use and powerful. Such **technology** easily produces graphs of most functions encountered in this book. We assume you know how to use a graphing utility.
- Graphing utilities, however, are not infallible. Therefore, you should also strive to master **analytical methods** (pencil-and-paper methods) in order to analyze functions and make accurate graphs by hand. Analytical methods rely heavily on calculus and are presented throughout this book.

The important message is this: Both technology and analytical methods are essential and must be used together in an integrated way to produce accurate graphs.

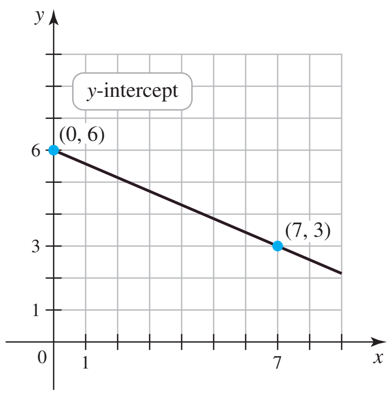


Figure 1.19

Linear Functions One form of the equation of a line (see Appendix A) is $y = mx + b$, where m and b are constants. Therefore, the function $f(x) = mx + b$ has a straight-line graph and is called a **linear function**.

EXAMPLE 1 **Linear functions and their graphs** Determine the function represented by the line in Figure 1.19.

SOLUTION From the graph, we see that the y-intercept is $(0, 6)$. Using the points $(0, 6)$ and $(7, 3)$, the slope of the line is

$$m = \frac{3-6}{7-0} = -\frac{3}{7}.$$

Therefore, the line is described by the function $f(x) = -3x/7 + 6$.

Related Exercises 11–14 ◄

EXAMPLE 2 **Demand function for pizzas** After studying sales for several months, the owner of a pizza chain knows that the number of two-topping pizzas sold in a week (called the *demand*) decreases as the price increases. Specifically, her data indicate that at a price of \$14 per pizza, an average of 400 pizzas are sold per week, while at a price of \$17 per pizza, an average of 250 pizzas are sold per week. Assume that the demand d is a *linear* function of the price p.

a. Find the constants m and b in the demand function $d = f(p) = mp + b$. Then graph f.

b. According to this model, how many pizzas (on average) are sold per week at a price of \$20?

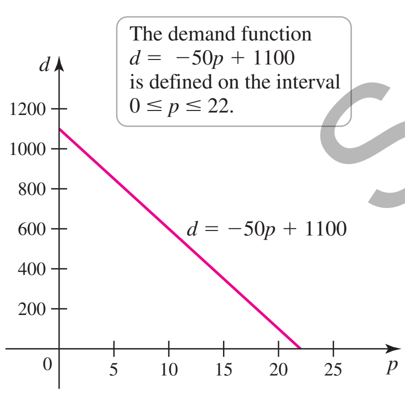


Figure 1.20

➤ The units of the slope have meaning: For every dollar the price is reduced, an average of 50 more pizzas can be sold.

SOLUTION

a. Two points on the graph of the demand function are given: $(p, d) = (14, 400)$ and $(17, 250)$. Therefore, the slope of the demand line is

$$m = \frac{400 - 250}{14 - 17} = -50 \text{ pizzas per dollar.}$$

It follows that the equation of the linear demand function is

$$d - 250 = -50(p - 17).$$

Expressing d as a function of p, we have $d = f(p) = -50p + 1100$ (Figure 1.20).

b. Using the demand function with a price of \$20, the average number of pizzas that could be sold per week is $f(20) = 100$.

Related Exercises 15–18 ◄

Piecewise Functions A function may have different definitions on different parts of its domain. For example, income tax is levied in tax brackets that have different tax rates. Functions that have different definitions on different parts of their domain are called **piecewise functions**. If all the pieces are linear, the function is **piecewise linear**. Here are some examples.

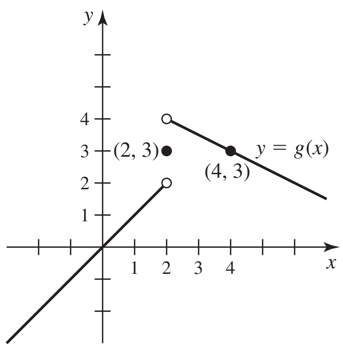


Figure 1.21

EXAMPLE 3 **Defining a piecewise function** The graph of a piecewise linear function g is shown in Figure 1.21. Find a formula for the function.

SOLUTION For $x < 2$, the graph is linear with a slope of 1 and a y-intercept of $(0, 0)$; its equation is $y = x$. For $x > 2$, the slope of the line is $-\frac{1}{2}$ and it passes through $(4, 3)$; so an equation of this piece of the function is

$$y - 3 = -\frac{1}{2}(x - 4) \quad \text{or} \quad y = -\frac{1}{2}x + 5.$$

For $x = 2$, we have $g(2) = 3$. Therefore,

$$g(x) = \begin{cases} x & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ -\frac{1}{2}x + 5 & \text{if } x > 2. \end{cases}$$

Related Exercises 19–22 ◄

EXAMPLE 4 **Graphing piecewise functions** Graph the following functions.

a. $f(x) = \begin{cases} \dfrac{x^2 - 5x + 6}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

b. $f(x) = |x|$, the **absolute value** function

SOLUTION

a. The function f is simplified by factoring and then canceling $x - 2$, assuming $x \neq 2$:

$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x - 2)(x - 3)}{x - 2} = x - 3.$$

Therefore, the graph of f is identical to the graph of the line $y = x - 3$ when $x \neq 2$. We are given that $f(2) = 1$ (Figure 1.22).

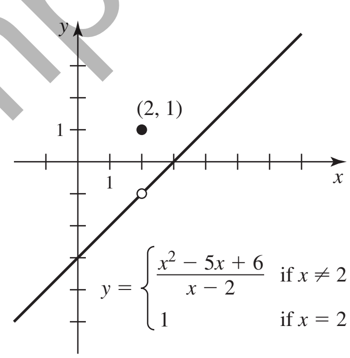


Figure 1.22

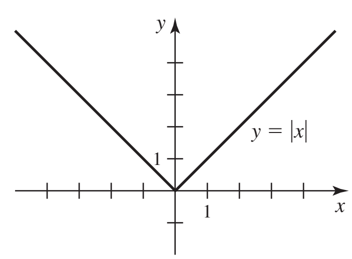


Figure 1.23

b. The absolute value of a real number is defined as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Graphing $y = -x$, for $x < 0$, and $y = x$, for $x \geq 0$, produces the graph in Figure 1.23.

Related Exercises 23–28 ◄

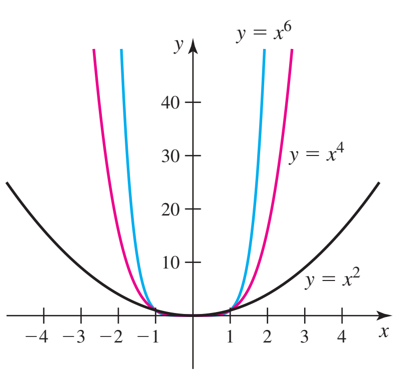


Figure 1.24

Power Functions Power functions are a special case of polynomials; they have the form $f(x) = x^n$, where n is a positive integer. When n is an even integer, the function values are nonnegative and the graph passes through the origin, opening upward (Figure 1.24). For

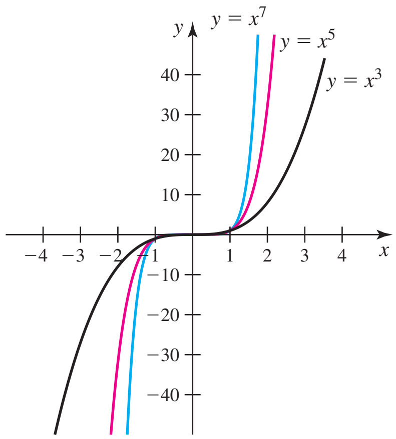


Figure 1.25

odd integers, the power function $f(x) = x^n$ has values that are positive when x is positive and negative when x is negative (Figure 1.25).

QUICK CHECK 2 What is the range of $f(x) = x^7$? What is the range of $f(x) = x^8$? ◄

Root Functions Root functions are a special case of algebraic functions; they have the form $f(x) = x^{1/n}$, where $n > 1$ is a positive integer. Notice that when n is even (square roots, fourth roots, and so forth), the domain and range consist of nonnegative numbers. Their graphs begin steeply at the origin and flatten out as x increases (Figure 1.26).

By contrast, the odd root functions (cube roots, fifth roots, and so forth) are defined for all real values of x and their range is all real numbers. Their graphs pass through the origin, open upward for $x < 0$ and downward for $x > 0$, and flatten out as x increases in magnitude (Figure 1.27).

➤ Recall that if n is a positive integer, then $x^{1/n}$ is the nth root of x; that is, $f(x) = x^{1/n} = \sqrt[n]{x}$.

QUICK CHECK 3 What are the domain and range of $f(x) = x^{1/7}$? What are the domain and range of $f(x) = x^{1/10}$? ◄

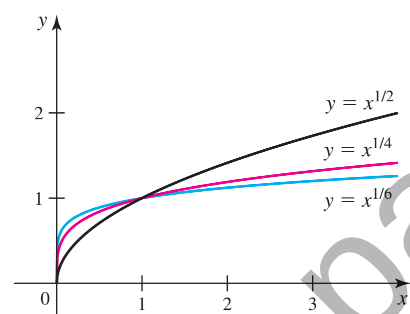


Figure 1.26

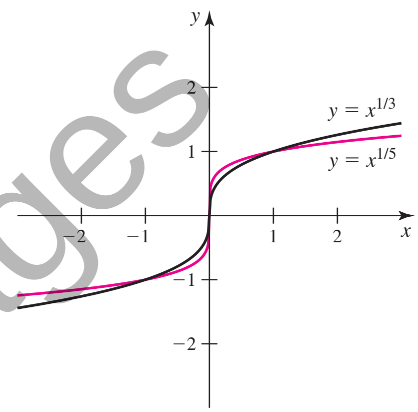


Figure 1.27

Rational Functions Rational functions appear frequently in this book, and much is said later about graphing rational functions. The following example illustrates how analysis and technology work together.

EXAMPLE 5 **Technology and analysis** Consider the rational function

$$f(x) = \frac{3x^3 - x - 1}{x^3 + 2x^2 - 6}.$$

a. What is the domain of f?

b. Find the roots (zeros) of f.

c. Graph the function using a graphing utility.

d. At what points does the function have peaks and valleys?

e. How does f behave as x grows large in magnitude?

SOLUTION

a. The domain consists of all real numbers except those at which the denominator is zero. A graphing utility shows that the denominator has one real zero at $x \approx 1.34$ and therefore, the domain of f is $\{x: x \neq 1.34\}$.

b. The roots of a rational function are the roots of the numerator, provided they are not also roots of the denominator. Using a graphing utility, the only real root of the numerator is $x \approx 0.85$.

c. After experimenting with the graphing window, a reasonable graph of f is obtained (Figure 1.28). At the point $x \approx 1.34$, where the denominator is zero, the function becomes large in magnitude and f has a *vertical asymptote*.

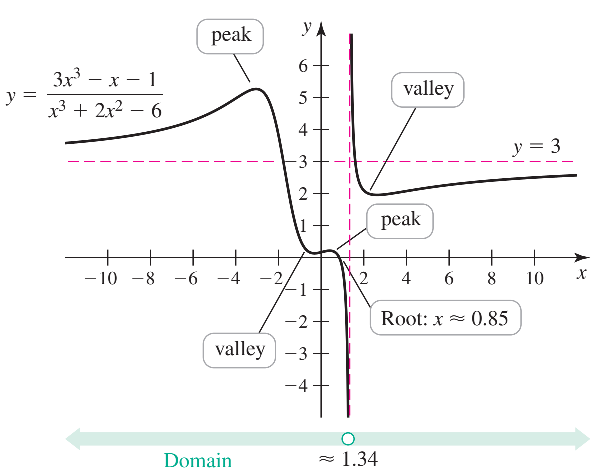


Figure 1.28

> In Chapter 3, we show how calculus is used to locate the local maximum and local minimum values of a function.

d. The function has two peaks (soon to be called *local maxima*), one near $x = -3.0$ and one near $x = 0.4$. The function also has two valleys (soon to be called *local minima*), one near $x = -0.3$ and one near $x = 2.6$.

e. By zooming out, it appears that as x increases in the positive direction, the graph approaches the *horizontal asymptote* $y = 3$ from below, and as x becomes large and negative, the graph approaches $y = 3$ from above.

Related Exercises 29–34 ◀

Using Tables

Sometimes functions do not originate as formulas or graphs; they may start as numbers or data. For example, suppose you do an experiment in which a marble is dropped into a cylinder filled with heavy oil and is allowed to fall freely. You measure the total distance d, in centimeters, that the marble falls at times $t = 0, 1, 2, 3, 4, 5, 6$, and 7 seconds after it is dropped (Table 1.1). The first step might be to plot the data points (Figure 1.29).

Table 1.1

t (s)	*d* (cm)
0	0
1	2
2	6
3	14
4	24
5	34
6	44
7	54

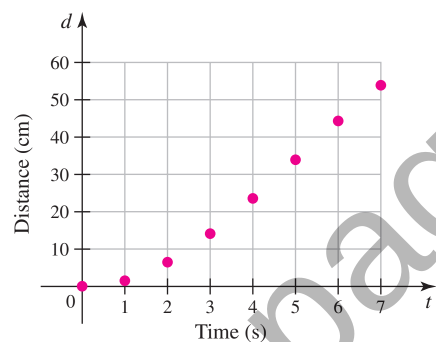


Figure 1.29

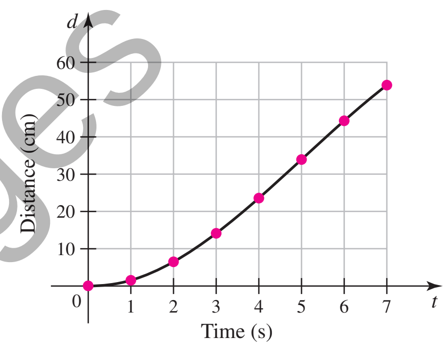


Figure 1.30

The data points suggest that there is a function $d = f(t)$ that gives the distance that the marble falls at *all* times of interest. Because the marble falls through the oil without abrupt changes, a smooth graph passing through the data points (Figure 1.30) is reasonable. Finding the best function that fits the data is a more difficult problem, which we discuss later in the text.

Using Words

Using words may be the least mathematical way to define functions, but it is often the way in which functions originate. Once a function is defined in words, it can often be tabulated, graphed, or expressed as a formula.

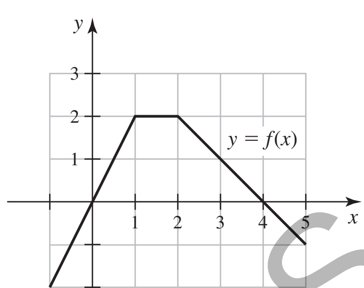


Figure 1.31

EXAMPLE 6 **A slope function** Let g be the **slope function** for a given function f. In words, this means that $g(x)$ is the slope of the curve $y = f(x)$ at the point $(x, f(x))$. Find and graph the slope function for the function f in Figure 1.31.

SOLUTION For $x < 1$, the slope of $y = f(x)$ is 2. The slope is 0 for $1 < x < 2$, and the slope is -1 for $x > 2$. At $x = 1$ and $x = 2$, the graph of f has a corner, so the slope is undefined at these points. Therefore, the domain of g is the set of all real numbers except $x = 1$ and $x = 2$, and the slope function (Figure 1.32) is defined by the piecewise function

$$g(x) = \begin{cases} 2 & \text{if } x < 1 \\ 0 & \text{if } 1 < x < 2 \\ -1 & \text{if } x > 2. \end{cases}$$

Related Exercises 35–38 ◀

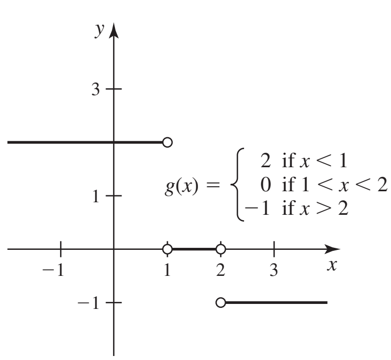


Figure 1.32

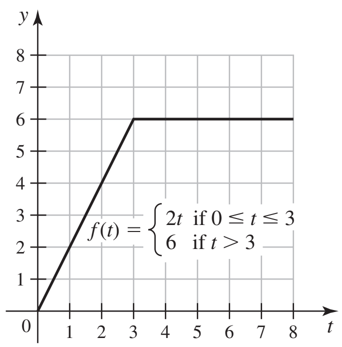


Figure 1.33

EXAMPLE 7 **An area function** Let A be an **area function** for a positive function f. In words, this means that $A(x)$ is the area of the region bounded by the graph of f and the t-axis from $t = 0$ to $t = x$. Consider the function (Figure 1.33)

$$f(t) = \begin{cases} 2t & \text{if } 0 \le t \le 3 \\ 6 & \text{if } t > 3. \end{cases}$$

a. Find $A(2)$ and $A(5)$.

b. Find a piecewise formula for the area function for f.

SOLUTION

a. The value of $A(2)$ is the area of the shaded region between the graph of f and the t-axis from $t = 0$ to $t = 2$ (Figure 1.34a). Using the formula for the area of a triangle,

$$A(2) = \frac{1}{2}(2)(4) = 4.$$

➤ Slope functions and area functions reappear in upcoming chapters and play an essential part in calculus.

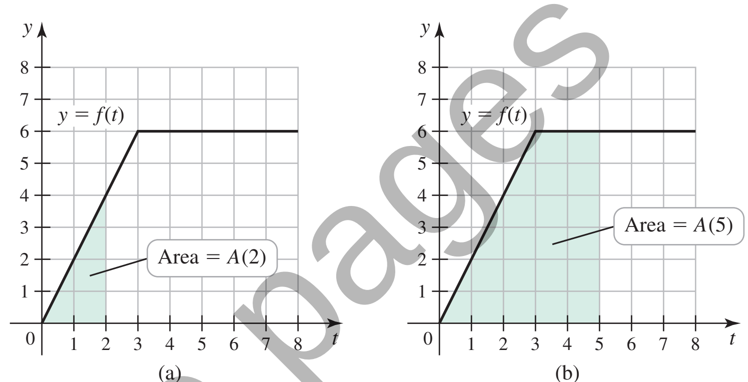


Figure 1.34

The value of $A(5)$ is the area of the shaded region between the graph of f and the t-axis on the interval $[0, 5]$ (Figure 1.34b). This area equals the area of the triangle whose base is the interval $[0, 3]$ plus the area of the rectangle whose base is the interval $[3, 5]$:

$$A(5) = \overbrace{\frac{1}{2}(3)(6)}^{\text{area of the triangle}} + \overbrace{(2)(6)}^{\text{area of the rectangle}} = 21.$$

b. For $0 \le x \le 3$ (Figure 1.35a), $A(x)$ is the area of the triangle whose base is the interval $[0, x]$. Because the height of the triangle at $t = x$ is $f(x)$,

$$A(x) = \frac{1}{2}xf(x) = \frac{1}{2}x\underbrace{(2x)}_{f(x)} = x^2.$$

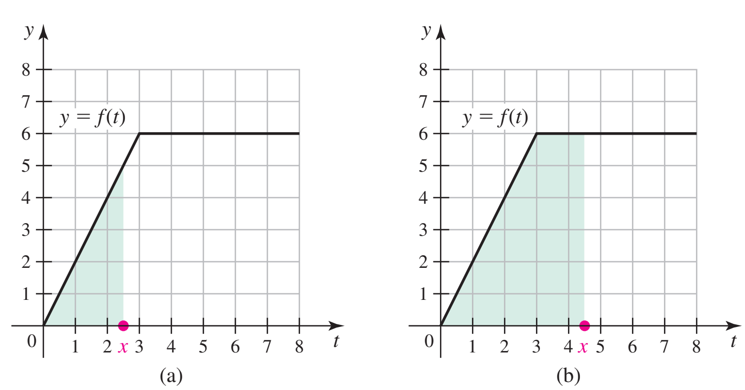


Figure 1.35

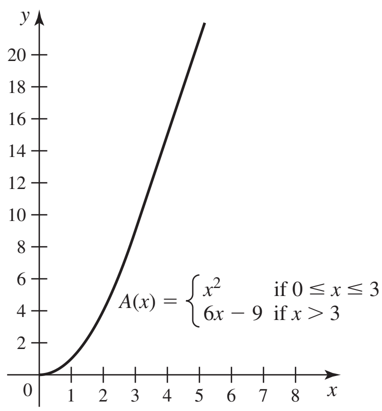


Figure 1.36

For $x > 3$ (Figure 1.35b), $A(x)$ is the area of the triangle on the interval $[0, 3]$ plus the area of the rectangle on the interval $[3, x]$:

$$A(x) = \overbrace{\frac{1}{2}(3)(6)}^{\text{area of the triangle}} + \overbrace{(x - 3)(6)}^{\text{area of the rectangle}} = 6x - 9.$$

Therefore, the area function A (Figure 1.36) has the piecewise definition

$$y = A(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 3 \\ 6x - 9 & \text{if } x > 3. \end{cases}$$

Related Exercises 39–42 ◂

Transformations of Functions and Graphs

There are several ways to transform the graph of a function to produce graphs of new functions. Four transformations are common: *shifts* in the x- and y-directions and *scalings* in the x- and y-directions. These transformations, summarized in Figures 1.37–1.42, can save time in graphing and visualizing functions.

The graph of $y = f(x) + d$ is the graph of $y = f(x)$ shifted vertically by d units (up if $d > 0$ and down if $d < 0$).

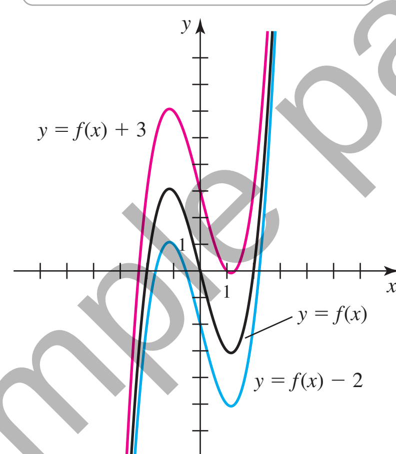


Figure 1.37

The graph of $y = f(x - b)$ is the graph of $y = f(x)$ shifted horizontally by b units (right if $b > 0$ and left if $b < 0$).

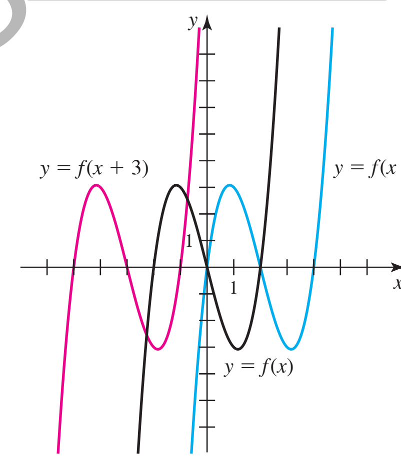


Figure 1.38

For $c > 0$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of c (wider if $0 < c < 1$ and narrower if $c > 1$).

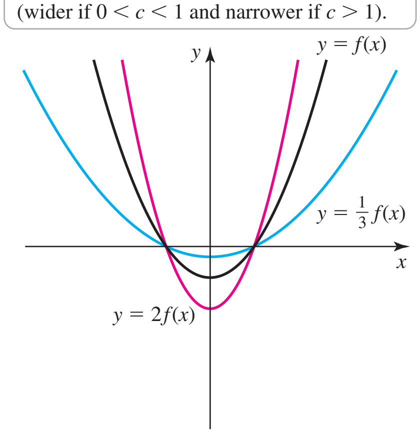


Figure 1.39

For $c < 0$, the graph of $y = cf(x)$ is the graph of $y = f(x)$ scaled vertically by a factor of $|c|$ and reflected across the x-axis (wider if $-1 < c < 0$ and narrower if $c < -1$).

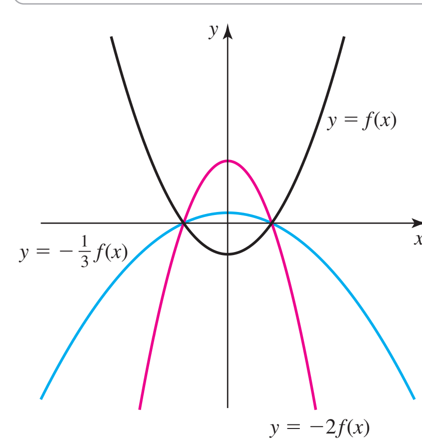


Figure 1.40

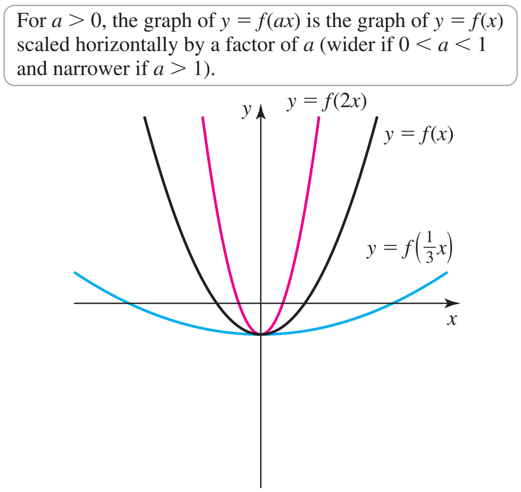


Figure 1.41

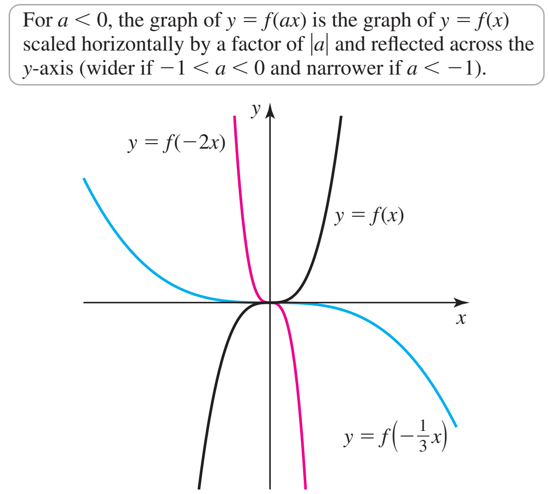


Figure 1.42

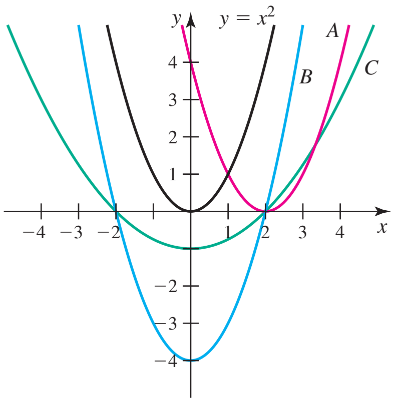


Figure 1.43

➤ You should verify that graph C also corresponds to a horizontal scaling and a vertical shift. It has the equation $y = f(ax) - 1$, where $a = \frac{1}{2}$.

EXAMPLE 8 **Shifting parabolas** The graphs A, B, and C in Figure 1.43 are obtained from the graph of $f(x) = x^2$ using shifts and scalings. Find the function that describes each graph.

SOLUTION

a. Graph A is the graph of f shifted to the right by 2 units. It represents the function

$$f(x - 2) = (x - 2)^2 = x^2 - 4x + 4.$$

b. Graph B is the graph of f shifted down by 4 units. It represents the function

$$f(x) - 4 = x^2 - 4.$$

c. Graph C is a wider version of the graph of f shifted down by 1 unit. Therefore, it represents $cf(x) - 1 = cx^2 - 1$, for some value of c, with $0 < c < 1$ (because the graph is widened). Using the fact that graph C passes through the points $(\pm 2, 0)$, we find that $c = \frac{1}{4}$. Therefore, the graph represents

$$y = \frac{1}{4}f(x) - 1 = \frac{1}{4}x^2 - 1.$$

Related Exercises 43–54 ◂

QUICK CHECK 4 How do you modify the graph of $f(x) = 1/x$ to produce the graph of $g(x) = 1/(x + 4)$? ◂

EXAMPLE 9 **Scaling and shifting** Graph $g(x) = |2x + 1|$.

SOLUTION We write the function as $g(x) = \left|2\left(x + \frac{1}{2}\right)\right|$. Letting $f(x) = |x|$, we have $g(x) = f\left(2\left(x + \frac{1}{2}\right)\right)$. Therefore, the graph of g is obtained by scaling (steepening) the graph of f horizontally and shifting it $\frac{1}{2}$ unit to the left (Figure 1.44).

➤ Note that we can also write $g(x) = 2\left|x + \frac{1}{2}\right|$, which means the graph of g may also be obtained by a vertical scaling and a horizontal shift.

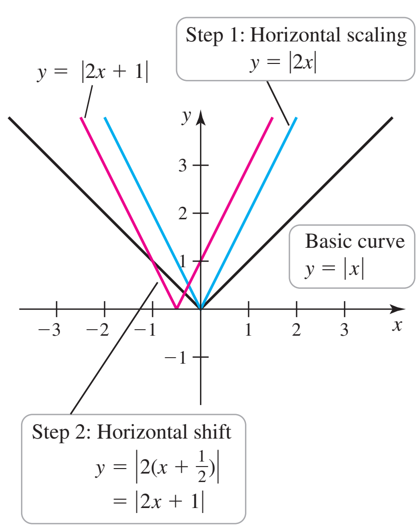


Figure 1.44

Related Exercises 43–54 ◂

SUMMARY Transformations

Given the real numbers a, b, c, and d and the function f, the graph of $y = cf(a(x - b)) + d$ can be obtained from the graph of $y = f(x)$ in the following steps.

$$y = f(x) \xrightarrow{\textit{horizontal scaling by a factor of } |a|} y = f(ax)$$

$$\xrightarrow{\textit{horizontal shift by } b \textit{ units}} y = f(a(x - b))$$

$$\xrightarrow{\textit{vertical scaling by a factor of } |c|} y = cf(a(x - b))$$

$$\xrightarrow{\textit{vertical shift by } d \textit{ units}} y = cf(a(x - b)) + d$$

SECTION 1.2 EXERCISES

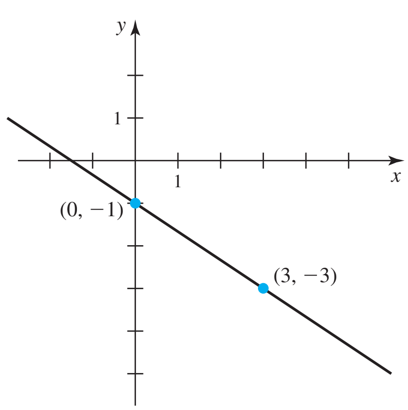
Review Questions

1. Give four ways that functions may be defined and represented.
2. What is the domain of a polynomial?
3. What is the domain of a rational function?
4. Describe what is meant by a piecewise linear function.
5. Sketch a graph of $y = x^5$.
6. Sketch a graph of $y = x^{1/5}$.
7. How do you obtain the graph of $y = f(x + 2)$ from the graph of $y = f(x)$?
8. How do you obtain the graph of $y = -3f(x)$ from the graph of $y = f(x)$?
9. How do you obtain the graph of $y = f(3x)$ from the graph of $y = f(x)$?
10. How do you obtain the graph of $y = 4(x + 3)^2 + 6$ from the graph of $y = x^2$?

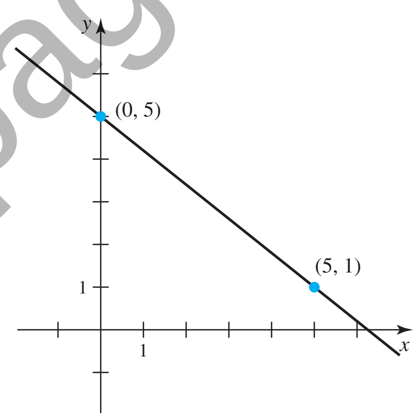
Basic Skills

11–12. Graphs of functions *Find the linear functions that correspond to the following graphs.*

11.



12.

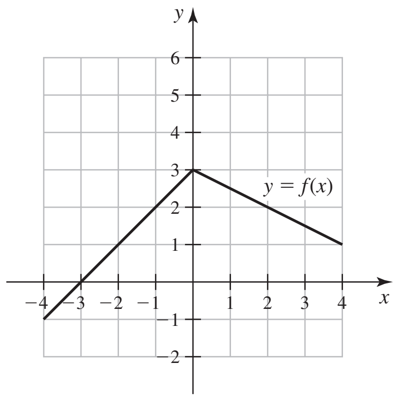


13. **Graph of a linear function** Find and graph the linear function that passes through the points $(1, 3)$ and $(2, 5)$.
14. **Graph of a linear function** Find and graph the linear function that passes through the points $(2, -3)$ and $(5, 0)$.
15. **Demand function** Sales records indicate that if Blu-ray players are priced at \$250, then a large store sells an average of 12 units per day. If they are priced at \$200, then the store sells an average of 15 units per day. Find and graph the linear demand function for Blu-ray sales. For what prices is the demand function defined?
16. **Fundraiser** The Biology Club plans to have a fundraiser for which \$8 tickets will be sold. The cost of room rental and refreshments is \$175. Find and graph the function $p = f(n)$ that gives the profit from the fundraiser when n tickets are sold. Notice that $f(0) = -\$175$; that is, the cost of room rental and refreshments must be paid regardless of how many tickets are sold. How many tickets must be sold to break even (zero profit)?
17. **Population function** The population of a small town was 500 in 2015 and is growing at a rate of 24 people per year. Find and graph the linear population function $p(t)$ that gives the population of the town t years after 2015. Then use this model to predict the population in 2030.

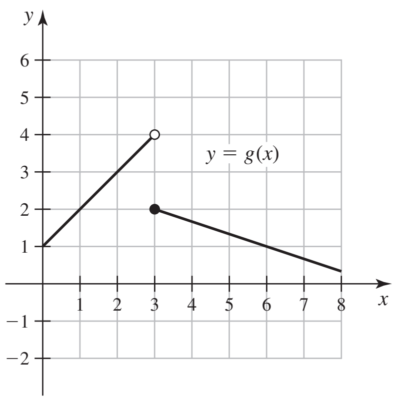
18. **Taxicab fees** A taxicab ride costs \$3.50 plus \$2.50 per mile. Let m be the distance (in miles) from the airport to a hotel. Find and graph the function $c(m)$ that represents the cost of taking a taxi from the airport to the hotel. Also determine how much it costs if the hotel is 9 miles from the airport.

19–20. Graphs of piecewise functions *Write a definition of the functions whose graphs are given.*

19.



20.



21. **Parking fees** Suppose that it costs 5¢ per minute to park at the airport with the rate dropping to 3¢ per minute after 9 P.M. Find and graph the cost function $c(t)$ for values of t satisfying $0 \le t \le 120$. Assume that t is the number of minutes after 8 P.M.

22. **Taxicab fees** A taxicab ride costs \$3.50 plus \$2.50 per mile for the first 5 miles, with the rate dropping to \$1.50 per mile after the fifth mile. Let m be the distance (in miles) from the airport to a hotel. Find and graph the piecewise linear function $c(m)$ that represents the cost of taking a taxi from the airport to a hotel m miles away.

23–28. Piecewise linear functions *Graph the following functions.*

23. $f(x) = \begin{cases} \dfrac{x^2 - x}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

24. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$

25. $f(x) = \begin{cases} 3x - 1 & \text{if } x \le 0 \\ -2x + 1 & \text{if } x > 0 \end{cases}$

26. $f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \ge 1 \end{cases}$

27. $f(x) = \begin{cases} -2x - 1 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$

28. $f(x) = \begin{cases} 2x + 2 & \text{if } x < 0 \\ x + 2 & \text{if } 0 \le x \le 2 \\ 3 - x/2 & \text{if } x > 2 \end{cases}$

T **29–34. Graphs of functions**

a. *Use a graphing utility to produce a graph of the given function. Experiment with different windows to see how the graph changes on different scales. Sketch an accurate graph by hand after using the graphing utility.*
b. *Give the domain of the function.*
c. *Discuss interesting features of the function, such as peaks, valleys, and intercepts (as in Example 5).*

29. $f(x) = x^3 - 2x^2 + 6$

30. $f(x) = \sqrt[3]{2x^2 - 8}$

31. $g(x) = \left| \dfrac{x^2 - 4}{x + 3} \right|$

32. $f(x) = \dfrac{\sqrt{3x^2 - 12}}{x + 1}$

33. $f(x) = 3 - |2x - 1|$

34. $f(x) = \begin{cases} \dfrac{|x - 1|}{x - 1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

35–38. Slope functions *Determine the slope function for the following functions.*

35. $f(x) = 2x + 1$

36. $f(x) = |x|$

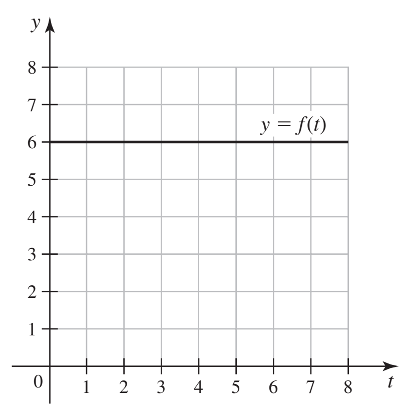
37. Use the figure for Exercise 19.

38. Use the figure for Exercise 20.

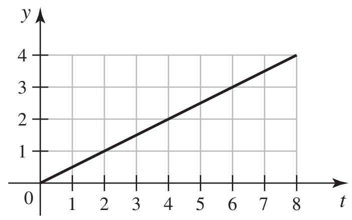
39–42. Area functions *Let $A(x)$ be the area of the region bounded by the t-axis and the graph of $y = f(t)$ from $t = 0$ to $t = x$. Consider the following functions and graphs.*

a. *Find $A(2)$.*
b. *Find $A(6)$.*
c. *Find a formula for $A(x)$.*

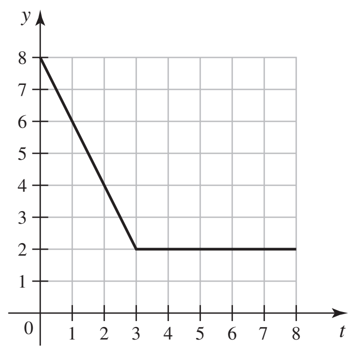
39. $f(t) = 6$



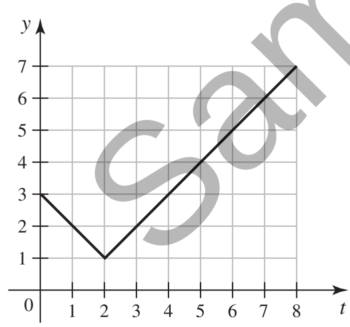
40. $f(t) = \dfrac{t}{2}$



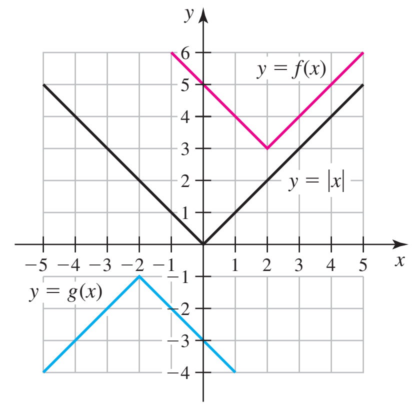
41. $f(t) = \begin{cases} -2t + 8 & \text{if } t \le 3 \\ 2 & \text{if } t > 3 \end{cases}$



42. $f(t) = |t - 2| + 1$

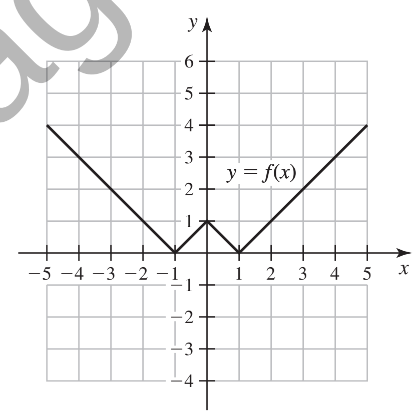


43. Transformations of $y = |x|$ The functions f and g in the figure are obtained by vertical and horizontal shifts and scalings of $y = |x|$. Find formulas for f and g. Verify your answers with a graphing utility.



44. Transformations Use the graph of f in the figure to plot the following functions.

a. $y = -f(x)$ **b.** $y = f(x + 2)$
c. $y = f(x - 2)$ **d.** $y = f(2x)$
e. $y = f(x - 1) + 2$ **f.** $y = 2f(x)$



45. Transformations of $f(x) = x^2$ Use shifts and scalings to transform the graph of $f(x) = x^2$ into the graph of g. Use a graphing utility to check your work.

a. $g(x) = f(x - 3)$ **b.** $g(x) = f(2x - 4)$
c. $g(x) = -3f(x - 2) + 4$ **d.** $g(x) = 6f\left(\dfrac{x - 2}{3}\right) + 1$

46. Transformations of $f(x) = \sqrt{x}$ Use shifts and scalings to transform the graph of $f(x) = \sqrt{x}$ into the graph of g. Use a graphing utility to check your work.

a. $g(x) = f(x + 4)$ **b.** $g(x) = 2f(2x - 1)$
c. $g(x) = \sqrt{x - 1}$ **d.** $g(x) = 3\sqrt{x - 1} - 5$

47–54. Shifting and scaling *Use shifts and scalings to graph the given functions. Then check your work with a graphing utility. Be sure to identify an original function on which the shifts and scalings are performed.*

47. $f(x) = (x - 2)^2 + 1$

48. $f(x) = x^2 - 2x + 3$ (*Hint:* Complete the square first.)

49. $g(x) = -3x^2$

50. $g(x) = 2x^3 - 1$

51. $g(x) = 2(x + 3)^2$

52. $p(x) = x^2 + 3x - 5$

53. $h(x) = -4x^2 - 4x + 12$

54. $h(x) = |3x - 6| + 1$

Further Explorations

55. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. All polynomials are rational functions, but not all rational functions are polynomials.
b. If f is a linear polynomial, then $f \circ f$ is a quadratic polynomial.
c. If f and g are polynomials, then the degrees of $f \circ g$ and $g \circ f$ are equal.
d. To graph $g(x) = f(x + 2)$, shift the graph of f 2 units to the right.

56–57. Intersection problems *Use analytical methods to find the following points of intersection. Use a graphing utility to check your work.*

56. Find the point(s) of intersection of the parabola $y = x^2 + 2$ and the line $y = x + 4$.

57. Find the point(s) of intersection of the parabolas $y = x^2$ and $y = -x^2 + 8x$.

58–59. Functions from tables *Find a simple function that fits the data in the tables.*

58.

x	y
−1	0
0	1
1	2
2	3
3	4

59.

x	y
0	−1
1	0
4	1
9	2
16	3

60–63. Functions from words *Find a formula for a function describing the given situation. Graph the function and give a domain that makes sense for the problem. Recall that with constant speed, distance = speed · time elapsed.*

60. A function $y = f(x)$ such that y is 1 less than the cube of x

61. Two cars leave a junction at the same time, one traveling north at 30 mi/hr, the other one traveling east at 60 mi/hr. The function $s(t)$ is the distance between the cars t hours after they leave the junction.

62. A function $y = f(x)$ such that if you ride a bike for 50 mi at x miles per hour, you arrive at your destination in y hours

63. A function $y = f(x)$ such that if your car gets 32 mi/gal and gasoline costs \$$x$/gallon, then \$100 is the cost of taking a y-mile trip

64. Floor function The floor function, or greatest integer function, $f(x) = \lfloor x \rfloor$, gives the greatest integer less than or equal to x. Graph the floor function, for $-3 \le x \le 3$.

65. Ceiling function The ceiling function, or smallest integer function, $f(x) = \lceil x \rceil$, gives the smallest integer greater than or equal to x. Graph the ceiling function, for $-3 \le x \le 3$.

66. Sawtooth wave Graph the sawtooth wave defined by

$$f(x) = \begin{cases} & \vdots \\ x + 1 & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 3 \\ & \vdots \end{cases}$$

67. Square wave Graph the square wave defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 2 \\ 1 & \text{if } 2 \le x < 3 \\ & \vdots \end{cases}$$

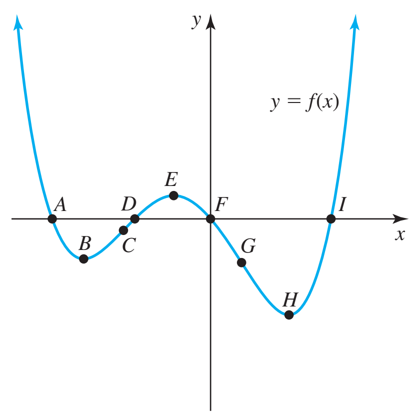
68–70. Roots and powers *Make a sketch of the given pairs of functions. Be sure to draw the graphs accurately relative to each other.*

68. $y = x^4$ and $y = x^6$

69. $y = x^3$ and $y = x^7$

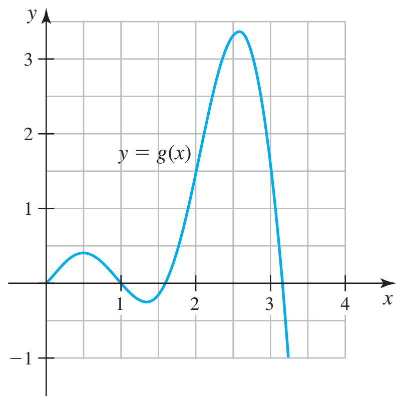
70. $y = x^{1/3}$ and $y = x^{1/5}$

71. Features of a graph Consider the graph of the function f shown in the figure. Answer the following questions by referring to the points A–I.



a. Which points correspond to the roots (zeros) of f?
b. Which points on the graph correspond to high points or peaks (soon to be called *local maximum* values of f)?
c. Which points on the graph correspond to low points or valleys (soon to be called *local minimum* values of f)?
d. As you move along the curve in the positive x-direction, at which point is the graph rising most rapidly?
e. As you move along the curve in the positive x-direction, at which point is the graph falling most rapidly?

72. Features of a graph Consider the graph of the function g shown in the figure.



a. Give the approximate roots (zeros) of g.
b. Give the approximate coordinates of the high points or peaks (soon to be called *local maximum* values of f).
c. Give the approximate coordinates of the low points or valleys (soon to be called *local minimum* values of f).
d. Imagine moving along the curve in the positive x-direction on the interval $[0, 3]$. Give the approximate coordinates of the point at which the graph is rising most rapidly.
e. Imagine moving along the curve in the positive x-direction on the interval $[0, 3]$. Give the approximate coordinates of the point at which the graph is falling most rapidly.

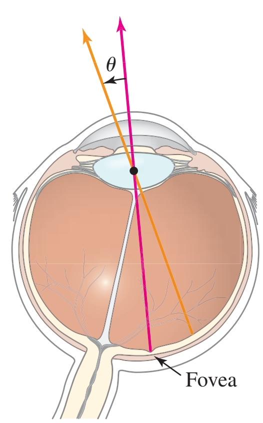
Applications

T **73. Relative acuity of the human eye** The **fovea centralis** (or **fovea**) is responsible for the sharp central vision that humans use for reading and other detail-oriented eyesight. The relative acuity of a human eye, which measures the sharpness of vision, is modeled by the function

$$R(\theta) = \frac{0.568}{0.331|\theta| + 0.568},$$

where θ (in degrees) is the angular deviation of the line of sight from the center of the fovea (see figure).

a. Graph R, for $-15 \leq \theta \leq 15$.
b. For what value of θ is R maximized? What does this fact indicate about our eyesight?
c. For what values of θ do we maintain at least 90% of our maximum relative acuity? (*Source: The Journal of Experimental Biology*, 203, Dec 2000)



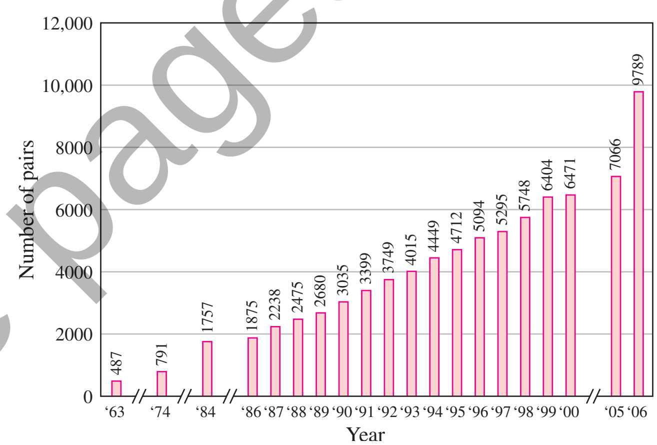
74. Tennis probabilities Suppose the probability of a server winning any given point in a tennis match is a constant p, with $0 \leq p \leq 1$. Then the probability of the server winning a game when serving from deuce is

$$f(p) = \frac{p^2}{1 - 2p(1 - p)}.$$

a. Evaluate $f(0.75)$ and interpret the result.
b. Evaluate $f(0.25)$ and interpret the result.
(*Source: The College Mathematics Journal* 38, 1, Jan 2007).

75. Bald eagle population Since DDT was banned and the Endangered Species Act was passed in 1973, the number of bald eagles in the United States has increased dramatically (see figure). In the lower 48 states, the number of breeding pairs of bald eagles increased at a nearly linear rate from 1875 pairs in 1986 to 6471 pairs in 2000.

a. Use the data points for 1986 and 2000 to find a linear function p that models the number of breeding pairs from 1986 to 2000 $(0 \leq t \leq 14)$.
b. Using the function in part (a), approximately how many breeding pairs were in the lower 48 states in 1995?



(*Source:* U.S. Fish and Wildlife Service)

76. Temperature scales

a. Find the linear function $C = f(F)$ that gives the reading on the Celsius temperature scale corresponding to a reading on the Fahrenheit scale. Use the facts that $C = 0$ when $F = 32$ (freezing point) and $C = 100$ when $F = 212$ (boiling point).
b. At what temperature are the Celsius and Fahrenheit readings equal?

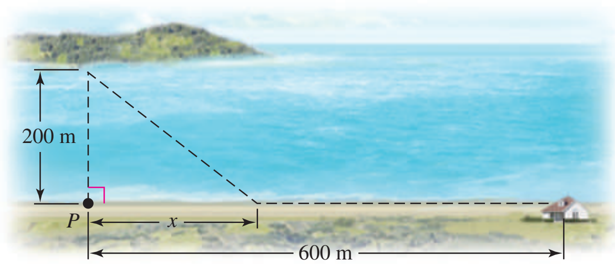
77. Automobile lease vs. purchase A car dealer offers a purchase option and a lease option on all new cars. Suppose you are interested in a car that can be bought outright for \$25,000 or leased for a start-up fee of \$1200 plus monthly payments of \$350.

a. Find the linear function $y = f(m)$ that gives the total amount you have paid on the lease option after m months.
b. With the lease option, after a 48-month (4-year) term, the car has a residual value of \$10,000, which is the amount that you could pay to purchase the car. Assuming no other costs, should you lease or buy?

78. Surface area of a sphere The surface area of a sphere of radius r is $S = 4\pi r^2$. Solve for r in terms of S and graph the radius function for $S \geq 0$.

T **79. Volume of a spherical cap** A single slice through a sphere of radius r produces a *cap* of the sphere. If the thickness of the cap is h, then its volume is $V = \frac{1}{3}\pi h^2(3r - h)$. Graph the volume as a function of h for a sphere of radius 1. For what values of h does this function make sense?

T **80. Walking and rowing** Kelly has finished a picnic on an island that is 200 m off shore (see figure). She wants to return to a beach house that is 600 m from the point P on the shore closest to the island. She plans to row a boat to a point on shore x meters from P and then jog along the (straight) shore to the house.



a. Let $d(x)$ be the total length of her trip as a function of x. Find and graph this function.
b. Suppose that Kelly can row at 2 m/s and jog at 4 m/s. Let $T(x)$ be the total time for her trip as a function of x. Find and graph $y = T(x)$.
c. Based on your graph in part (b), estimate the point on the shore at which Kelly should land to minimize the total time of her trip. What is that minimum time?

T **81. Optimal boxes** Imagine a lidless box with height h and a square base whose sides have length x. The box must have a volume of 125 ft^3.

a. Find and graph the function $S(x)$ that gives the surface area of the box, for all values of $x > 0$.
b. Based on your graph in part (a), estimate the value of x that produces the box with a minimum surface area.

Additional Exercises

82. Composition of polynomials Let f be an nth-degree polynomial and let g be an mth-degree polynomial. What is the degree of the following polynomials?

a. $f \cdot f$ **b.** $f \circ f$ **c.** $f \cdot g$ **d.** $f \circ g$

83. Parabola vertex property Prove that if a parabola crosses the x-axis twice, the x-coordinate of the vertex of the parabola is halfway between the x-intercepts.

84. Parabola properties Consider the general quadratic function $f(x) = ax^2 + bx + c$, with $a \neq 0$.

a. Find the coordinates of the vertex in terms of a, b, and c.
b. Find the conditions on a, b, and c that guarantee that the graph of f crosses the x-axis twice.

T **85. Factorial function** The factorial function is defined for positive integers as $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

a. Make a table of the factorial function, for $n = 1, 2, 3, 4, 5$.
b. Graph these data points and then connect them with a smooth curve.
c. What is the least value of n for which $n! > 10^6$?

T **86. Sum of integers** Let $S(n) = 1 + 2 + \cdots + n$, where n is a positive integer. It can be shown that $S(n) = n(n+1)/2$.

a. Make a table of $S(n)$, for $n = 1, 2, \ldots, 10$.
b. How would you describe the domain of this function?
c. What is the least value of n for which $S(n) > 1000$?

T **87. Sum of squared integers** Let $T(n) = 1^2 + 2^2 + \cdots + n^2$, where n is a positive integer. It can be shown that $T(n) = n(n+1)(2n+1)/6$.

a. Make a table of $T(n)$, for $n = 1, 2, \ldots, 10$.
b. How would you describe the domain of this function?
c. What is the least value of n for which $T(n) > 1000$?

QUICK CHECK ANSWERS

1. Yes; no **2.** $(-\infty, \infty)$; $[0, \infty)$ **3.** Domain and range are $(-\infty, \infty)$. Domain and range are $[0, \infty)$. **4.** Shift the graph of f horizontally 4 units to the left. ◄

1.3 Inverse, Exponential, and Logarithmic Functions

Exponential functions are fundamental to all of mathematics. Many processes in the world around us are modeled by *exponential functions*—they appear in finance, medicine, ecology, biology, economics, anthropology, and physics (among other disciplines). Every exponential function has an inverse function, which is a member of the family of *logarithmic functions*, also discussed in this section.

Exponential Functions

Exponential functions have the form $f(x) = b^x$, where the base $b \neq 1$ is a positive real number. An important question arises immediately: For what values of x can b^x be evaluated? We certainly know how to compute b^x when x is an integer. For example, $2^3 = 8$ and $2^{-4} = 1/2^4 = 1/16$. When x is rational, the numerator and denominator are interpreted as a power and root, respectively:

$$16^{3/4} = 16^{\overset{\text{power}}{3}/\underset{\text{root}}{4}} = (\underbrace{\sqrt[4]{16}}_{2})^3 = 8.$$

➤ $16^{3/4}$ can also be computed as $\sqrt[4]{16^3} = \sqrt[4]{4096} = 8$.

> **Exponent Rules**
>
> For any base $b > 0$ and real numbers x and y, the following relations hold:
>
> **E1.** $b^x b^y = b^{x+y}$
>
> **E2.** $\dfrac{b^x}{b^y} = b^{x-y}$ $\left(\text{which includes } \dfrac{1}{b^y} = b^{-y}\right)$
>
> **E3.** $(b^x)^y = b^{xy}$
>
> **E4.** $b^x > 0$, for all x

QUICK CHECK 1 Is it possible to raise a positive number b to a power and obtain a negative number? Is it possible to obtain zero?

QUICK CHECK 2 Explain why $f(x) = \left(\dfrac{1}{3}\right)^x$ is a decreasing function.

But what happens when x is irrational? For example, how should 2^π be understood? Your calculator provides an approximation to 2^π, but where does the approximation come from? These questions will be answered eventually. For now, we assume that b^x can be defined for all real numbers x and that it can be approximated as closely as desired by using rational numbers as close to x as needed. In Section 6.8, we prove that the domain of an exponential function is all real numbers.

Properties of Exponential Functions $f(x) = b^x$

1. Because b^x is defined for all real numbers, the domain of f is $\{x: -\infty < x < \infty\}$. Because $b^x > 0$ for all values of x, the range of f is $\{y: 0 < y < \infty\}$.
2. For all $b > 0$, $b^0 = 1$, and therefore $f(0) = 1$.
3. If $b > 1$, then f is an increasing function of x (Figure 1.45). For example, if $b = 2$, then $2^x > 2^y$ whenever $x > y$.
4. If $0 < b < 1$, then f is a decreasing function of x. For example, if $b = \frac{1}{2}$,

$$f(x) = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} = 2^{-x},$$

and because 2^x increases with x, 2^{-x} decreases with x (Figure 1.46).

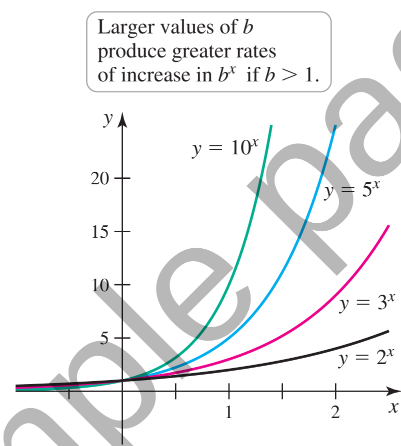


Figure 1.45

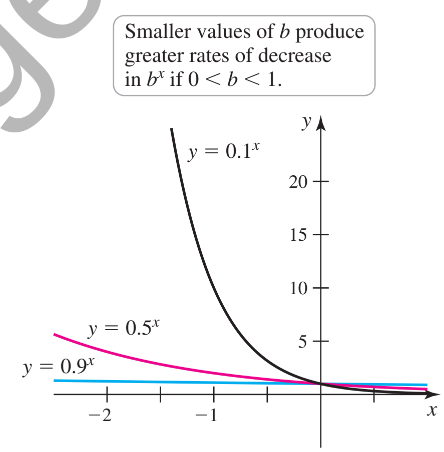


Figure 1.46

> The notation e was proposed by the Swiss mathematician Leonhard Euler (pronounced *oiler*) (1707–1783).

The Natural Exponential Function One of the bases used for exponential functions is special. For reasons that will become evident in upcoming chapters, the special base is e, one of the fundamental constants of mathematics. It is an irrational number with a value of $e = 2.718281828459\ldots$.

DEFINITION The Natural Exponential Function

The **natural exponential function** is $f(x) = e^x$, which has the base $e = 2.718281828459\ldots$.

The base e gives an exponential function that has a valuable property. As shown in Figure 1.47a, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$ (because $2 < e < 3$). At every point on the graph of $y = e^x$, it is possible to draw a *tangent line* (discussed in Chapters 2 and 3) that touches the graph only at that point. The natural exponential function is the only exponential function with the property that the slope of the tangent line at $x = 0$ is 1 (Figure 1.47b); therefore, e^x has both value and slope equal to 1 at $x = 0$. This property—minor as it may seem—leads to many simplifications when we do calculus with exponential functions.

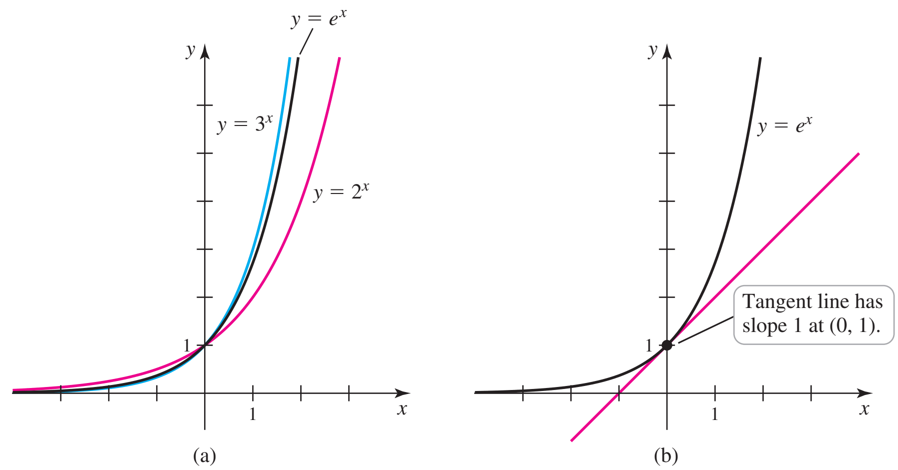


Figure 1.47

Inverse Functions

Consider the linear function $f(x) = 2x$, which takes any value of x and doubles it. The function that reverses this process by taking any value of $f(x) = 2x$ and mapping it back to x is called the *inverse function* of f, denoted f^{-1}. In this case, the inverse function is $f^{-1}(x) = x/2$. The effect of applying these two functions in succession looks like this:

$$x \xrightarrow{f} 2x \xrightarrow{f^{-1}} x$$

We now generalize this idea.

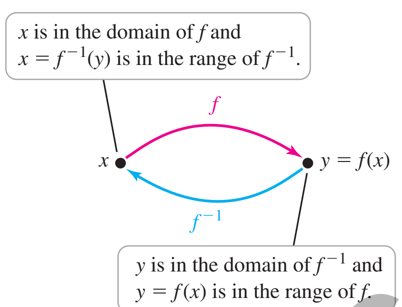


Figure 1.48

➤ The notation f^{-1} for the inverse can be confusing. The inverse is not the reciprocal; that is, $f^{-1}(x)$ is not $1/f(x) = (f(x))^{-1}$. We adopt the common convention of using simply *inverse* to mean *inverse function*.

DEFINITION Inverse Function

Given a function f, its inverse (if it exists) is a function f^{-1} such that whenever $y = f(x)$, then $f^{-1}(y) = x$ (Figure 1.48).

QUICK CHECK 3 What is the inverse of $f(x) = \frac{1}{3}x$? What is the inverse of $f(x) = x - 7$? ◀

Because the inverse "undoes" the original function, if we start with a value of x, apply f to it, and then apply f^{-1} to the result, we recover the original value of x; that is,

$$f^{-1}(f(x)) = x.$$

$$\begin{tikzcd} x \arrow[r, "f"'] \arrow[rr, bend left] & y \arrow[r, "f^{-1}"'] & x \end{tikzcd}$$

Similarly, if we apply f^{-1} to a value of y and then apply f to the result, we recover the original value of y; that is,

$$f(f^{-1}(y)) = y.$$

$$\begin{tikzcd} y \arrow[r, "f^{-1}"'] \arrow[rr, bend left] & x \arrow[r, "f"'] & y \end{tikzcd}$$

One-to-One Functions We have defined the inverse of a function, but said nothing about when it exists. To ensure that f has an inverse on a domain, f must be *one-to-one* on that domain. This property means that every output of the function f must correspond to