NEW SENIOR MATHEMATICS

EXTENSION 2

FOR YEAR 12

THIRD EDITION

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CONTENTS

CHAPTER 1
Complex numbers

1.1 ARITHMETIC OF COMPLEX NUMBERS AND THE SOLUTION OF QUADRATIC EQUATIONS

The need for complex numbers

As society has developed over time, so has our need for a more comprehensive number system. Initially, the only numbers needed were the counting numbers (1, 2, 3, …). Later, people found a need for zero and for negative numbers, giving us the set of integers. Fractions and decimals gave us the set of rational numbers. With numbers such as $\sqrt{2}$ and π, the set of irrational numbers was developed. The rationals and the irrationals together form the set of real numbers.

These number systems have been developed by mathematicians to address new and different problems that have emerged.

For example, to solve different kinds of equations requires different kinds of numbers. Using only integers, you can solve equations such as $x + 5 = 2$ but you can't solve $5x = 2$. You need rational numbers for that. To solve $x^2 = 5$ you need irrational numbers.

There are other equations which can't be solved using any real numbers. The simplest example is $x^2 = -1$ or $x^2 + 1 = 0$. However, this equation can be solved by defining a number i such that $i^2 = -1$:

$$x^2 + 1 = 0$$

i.e. $x^2 - i^2 = 0$ where $i^2 = -1$

$(x - i)(x + i) = 0$ (difference of two squares)

$\therefore \quad x = i \quad \text{or} \quad x = -i$

Example 1

Solve the quadratic equation $x^2 - 4x + 13 = 0$.

Solution

Note that the discriminant $\Delta = b^2 - 4ac = 16 - 52 = -36$. Hence the quadratic equation has no real roots and the parabola $y = x^2 - 4x + 13$ is entirely above the x-axis.

However, you can find solutions using **complex numbers**, which are of the form $a + bi$ where a and b are real numbers.

Method 1 Using the quadratic formula

$$x^2 - 4x + 13 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{4 \pm \sqrt{-36}}{2}$$
$$x = 2 \pm 3\sqrt{-1}$$
$$x = 2 \pm 3i$$

Method 2 Completing the square

$$x^2 - 4x + 13 = 0$$
$$x^2 - 4x + 4 + 9 = 0$$
$$(x - 2)^2 = -9$$
$$x - 2 = \pm 3i$$
$$x = 2 \pm 3i$$

The complex number system

Any number z of the form $x + iy$, where x and y are real numbers, is called a complex number.

x is the **real part** of z, denoted by $\text{Re}(z) = x$, and y is called the **imaginary part** of z (although it is not literally 'imaginary' in the usual sense of that word), denoted by $\text{Im}(z) = y$.

(Note that you use a single letter z to denote the complex number $x + iy$, to emphasise that $x + iy$ is a single number despite being written as a sum of two parts.)

If the imaginary part of z is zero, i.e. $y = 0$, then z is purely real. This means that the set of real numbers is a subset of the set of complex numbers.

If the real part of z is zero, i.e. $x = 0$, then z is purely imaginary, e.g. $3i$ or $-i$.

The following definitions apply to complex numbers.

Equality

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal:

$$a + bi = c + di \qquad \text{if and only if} \qquad a = c \quad \text{and} \quad b = d$$

The 'if and only if' in the above statement means that the statement applies forwards and backwards, in other words it is two statements in one.

If $a + bi = c + di$, then $a = c$ and $b = d$.

If $a = c$ and $b = d$, then $a + bi = c + di$.

Addition and subtraction

If $z = z_1 \pm z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z = (x_1 + x_2) \pm i(y_1 + y_2)$.

Multiplication

If $z = z_1 \times z_2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$\begin{aligned} z &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i^2y_1y_2 + ix_1y_2 + ix_2y_1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

The conjugate of a complex number

If $z = x + iy$, then the **conjugate** of z is $\overline{z} = x - iy$. (This is similar to the conjugate of a surd.)

Note that the product of a complex number and its conjugate is a real number:

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2 \end{aligned}$$

Division

To calculate the division $z = \dfrac{z_1}{z_2}$, multiply the numerator and denominator by the conjugate of z_2. This **realises** the denominator, i.e. makes the denominator real. This is similar to how you rationalise a denominator when dividing surds.

Example 2

If $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, find:

(a) $z_1 + z_2$ **(b)** $z_1 - z_2$ **(c)** $z_1 \times z_2$ **(d)** $z_2\overline{z}_2$ **(e)** $z_1^{\,2}$ **(f)** $z_1 \div z_2$

Solution

(a) $z_1 + z_2 = 2 + 3i + (-1 + 4i) = 1 + 7i$

(b) $z_1 - z_2 = 2 + 3i - (-1 + 4i) = 3 - i$

(c) $z_1 \times z_2 = (2 + 3i)(-1 + 4i) = -2 + 8i - 3i + 12i^2 = -2 + 5i - 12 = -14 + 5i$

(d) $z_2\overline{z}_2 = (-1+4i)(-1-4i) = (-1)^2 - 16i^2 = 1 + 16 = 17$

(e) $z_1{}^2 = (2+3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$

(f) $\dfrac{z_1}{z_2} = \dfrac{2+3i}{-1+4i} = \dfrac{(2+3i)}{(-1+4i)} \times \dfrac{(-1-4i)}{(-1-4i)} = \dfrac{-2-8i-3i-12i^2}{1-16i^2} = \dfrac{-2-11i+12}{1+16} = \dfrac{10-11i}{17} = \dfrac{10}{17} - \dfrac{11}{17}i$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 - 5i$, find:

(a) $z_1 + z_2$ (b) $z_2 - z_1$ (c) $z_1 + \overline{z}_1$ (d) $z_2 - \overline{z}_2$ (e) $\overline{z}_1 \times \overline{z}_2$ (f) $\dfrac{\overline{z}_2}{\overline{z}_1}$

Solution

(a) $z_1 + z_2 = 2 - 3i + (-4 - 5i) = -2 - 8i$

(b) $z_2 - z_1 = (-4 - 5i) - (2 - 3i) = -6 - 2i$

(c) $\overline{z}_1 = 2 + 3i.\quad z_1 + \overline{z}_1 = 2 - 3i + (2 + 3i) = 4$

(d) $\overline{z}_2 = -4 + 5i.\quad z_2 - \overline{z}_2 = -4 - 5i - (-4 + 5i) = -10i$

(e) $\overline{z}_1 \times \overline{z}_2 = (2+3i)(-4+5i) = -8 + 10i - 12i + 15i^2 = -8 - 15 - 2i = -23 - 2i$

(f) $\dfrac{\overline{z}_2}{\overline{z}_1} = \dfrac{-4+5i}{2+3i} = \dfrac{-4+5i}{2+3i} \times \dfrac{2-3i}{2-3i} = \dfrac{-8+12i+10i-15i^2}{4+9} = \dfrac{7+22i}{13} = \dfrac{7}{13} + \dfrac{22}{13}i$

Example 4

Express $z^3 + 64$ as the product of three linear factors. Hence find the three cube roots of -64.

Solution

$$\begin{aligned}
z^3 + 64 &= (z+4)(z^2 - 4z + 16) && \text{(sum of two cubes)}\\
&= (z+4)(z^2 - 4z + 4 + 12) && \text{(complete the square)}\\
&= (z+4)((z-2)^2 - 12i^2) && \text{(construct the difference of two squares)}\\
&= (z+4)((z-2)^2 - (2\sqrt{3}\,i)^2)\\
&= (z+4)(z - 2 - 2\sqrt{3}\,i)(z - 2 + 2\sqrt{3}\,i)
\end{aligned}$$

The cube roots of -64 are obtained from $z + 4 = 0$, $z - 2 - 2\sqrt{3}\,i = 0$, $z - 2 + 2\sqrt{3}\,i = 0$.

$\therefore$ The cube roots are -4, $2 - 2\sqrt{3}\,i$ and $2 + 2\sqrt{3}\,i$.

Square roots of a complex number

The general method for finding the square roots of a complex number is illustrated in the following example.

Example 5

Find the square roots of $3 + 4i$.

Solution

Let $z = x + iy$, where x, y are real, such that $z^2 = 3 + 4i$:

$$(x + iy)^2 = 3 + 4i$$

$$(x^2 - y^2) + 2xyi = 3 + 4i$$

Equating the real and imaginary parts of LHS and RHS:

$x^2 - y^2 = 3 \quad [1] \qquad\qquad 2xy = 4 \quad [2]$

From [2], $y = \frac{2}{x}$, then substitute into [1]:
$$x^2 - \frac{4}{x^2} = 3$$
$$x^4 - 3x^2 - 4 = 0$$
$$(x^2 - 4)(x^2 + 1) = 0$$
$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$

But x is real $\therefore\ x = \pm 2$ are the only solutions.

Substituting this into [2]: $y = \pm 1$

So the square roots of $3 + 4i$ are $2 + i$ and $-2 - i$, which can be written as $\pm(2 + i)$.

EXPLORING FURTHER

Arithmetic of complex numbers

Use technology to explore the arithmetic of complex numbers.

EXERCISE 1.1 ARITHMETIC OF COMPLEX NUMBERS AND THE SOLUTION OF QUADRATIC EQUATIONS

1 $i^5 = \ldots$

A 1 **B** -1 **C** i **D** $-i$

2 Solve the following equations.

(a) $z^2 + 9 = 0$ **(b)** $z^2 + 25 = 0$ **(c)** $z^2 + 2z + 17 = 0$
(d) $-z^2 + 2z - 5 = 0$ **(e)** $z^2 = 4z - 20$ **(f)** $-2z^2 + 2z - 13 = 0$
(g) $z^2 - z + 8 = 0$ **(h)** $z - 4 - z^2 = 0$

3 Simplify:

(a) i^3 **(b)** i^4 **(c)** i^6 **(d)** i^7 **(e)** i^8

4 If $z = 5 - 2i$, find:

(a) z^{-1} **(b)** $\overline{z}$ **(c)** $z\overline{z}$ **(d)** z^2 **(e)** $(z - \overline{z})^2$ **(f)** $\frac{z-1}{z-i}$

5 Simplify:

(a) $(3 + 5i) + (7 - 2i)$ **(b)** $(4 + 7i) - (-2 + 9i)$ **(c)** $(5 + 2i)(3 - 4i)$ **(d)** $(7 + 3i)(7 - 3i)$
(e) $(2 - 5i)^2$ **(f)** i^{17} **(g)** $(\sqrt{3} + 2i)(\sqrt{3} - 2i)$ **(h)** $\frac{1}{2+3i}$
(i) $\frac{8+5i}{4-3i}$ **(j)** $\frac{3i}{2+5i} + \frac{2}{2-5i}$ **(k)** $\frac{-8+3i}{-2-4i} - \frac{2+3i}{1+2i}$ **(l)** $\left(\frac{5+9i}{2-4i}\right)^2$

6 Find real numbers x and y such that:

(a) $(x + iy)(2 - 3i) = -13i$ **(b)** $(1 + i)x + (2 - 3i)y = 10$

7 If $z_1 = 3 + i$ and $z_2 = 2 - 3i$, find:

(a) $(z_1 - z_2)^2$ **(b)** $\overline{z_1} \times \overline{z_2}$ **(c)** $\overline{z_1 z_2}$ **(d)** $\frac{z_1 - z_2}{z_1 + z_2}$

8 Find the linear factors of the following expressions.

(a) $z^2 + 9$ **(b)** $z^2 + 36$ **(c)** $(z - 3)^2 + 16$ **(d)** $(2z + 3)^2 + 8$
(e) $z^2 + 2z + 26$ **(f)** $z^2 - 6z + 20$ **(g)** $2z^2 + 2z + 4$ **(h)** $z^3 + 1000$

9 Solve the equation: **(a)** $2z - 1 = (4 - i)^2$ **(b)** $\frac{z-2}{z} = 1 + i$

10 **(a)** Find the square roots of $-8+6i$.
(b) Hence solve $2z^2+(1-i)z+(1-i)=0$.
(c) Use your answer to part **(b)** to verify that the results for the sum of roots and for the product of roots of a quadratic equation are true when the coefficients and roots are complex numbers.

11 Find the square roots of the following: **(a)** $-8-6i$ **(b)** $-16i$ **(c)** $12+5i$

12 **(a)** Show that $\sqrt{3}-i$ is a root of the equation $z^3-(\sqrt{3}-i)z^2+9z-9\sqrt{3}+9i=0$.
(b) Find the other two solutions of the equation.
(c) Use your answer to part **(b)** to verify that the results for the sum of roots, for the sum of products of pairs of roots and for the product of roots of a cubic equation are true when the coefficients and roots are complex numbers.

13 Solve the following quadratic equations.
(a) $z^2-(3-2i)z+(1-3i)=0$ **(b)** $z^2-z+(4+2i)=0$
(c) $z^2-(2+2i)z+(-1+2i)=0$ **(d)** $z^2-(3+i)z+(2-3i)=0$

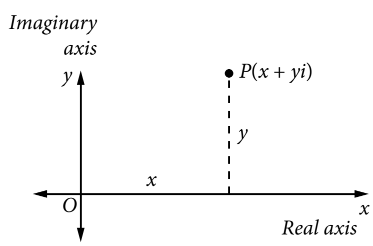
14 Let $z=a+ib$ where a, b are real. Prove that there are always two square roots of z except when $a=b=0$.

15 If $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, show that the following equations are true.
(a) $z_1+\overline{z_1}=2\times\mathrm{Re}(z_1)$ **(b)** $z_1-\overline{z_1}=2\times\mathrm{Im}(z_1)\times i$ **(c)** $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$
(d) $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$ **(e)** $\overline{z_1\times z_2}=\overline{z_1}\times\overline{z_2}$

16 **(a)** Express z^3+125 as the product of three linear factors.
Let w be one of the non-real complex roots of the equation $z^3+125=0$.
(b) Show that $w^2=5w-25$. **(c)** Hence simplify $(5w-25)^3$.

1.2 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A POINT

As the complex number $z=x+iy$ is composed of two parts, it can be considered as an ordered pair (x, y), and so complex numbers can be represented as points in a plane. Any complex number $z=x+iy$ can be represented by the point P with coordinates (x, y) in a number plane in which the x-axis is the 'real' axis and the y-axis is the 'imaginary' axis. This Cartesian representation of complex numbers is called the **Argand diagram**, after the French mathematician Jean-Robert Argand (1768–1822). The number plane on which Argand diagrams are mapped is called the **complex number plane**.



Geometrical addition and subtraction of complex numbers

Example 6

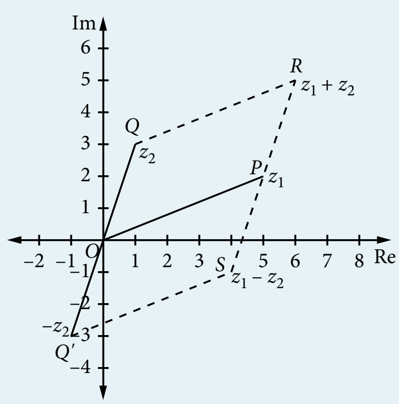
If $z_1=5+2i$ and $z_2=1+3i$, show z_1+z_2 and z_1-z_2 on an Argand diagram.

Solution

Algebraically, $z_1+z_2=6+5i$ and $z_1-z_2=4-i$.

On an Argand diagram, points P, Q and R represent z_1, z_2 and z_1+z_2 respectively. Note the location of R to complete the parallelogram $OPRQ$.

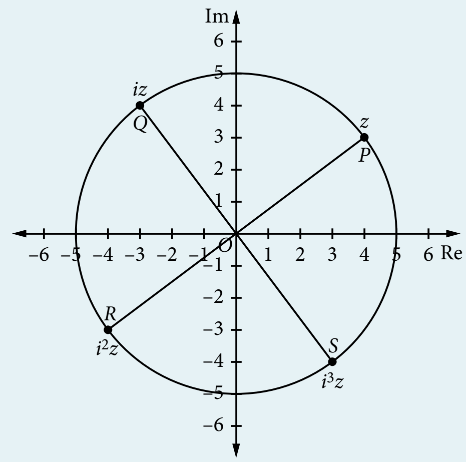
The diagram also shows points Q' and S, representing $-z_2$ and z_1-z_2 respectively. Note that z_1-z_2 has been calculated as $z_1+(-z_2)$. S completes the parallelogram $OPSQ'$.



Geometrical representation of multiplication by *i*

Example 7

If $z = 4 + 3i$, show iz, i^2z and i^3z on an Argand diagram.

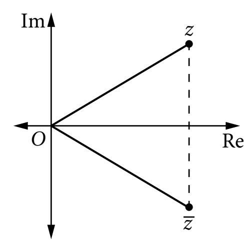


Solution

Algebraically: $iz = -3 + 4i$, $i^2z = -4 - 3i$ and $i^3z = 3 - 4i$.

The Argand diagram shows that each multiplication by i causes the point z to be rotated anticlockwise about the origin by $\frac{\pi}{2}$ (90°).

Geometrical representation of conjugates



The points that represent a pair of complex conjugates are reflections in the real axis. (This is because a number and its complex conjugate are the same except that the imaginary part has changed from negative to positive, or vice versa.)

If $z = x + iy$, then $\overline{z} = x - iy$. The sign of the imaginary part has changed, while the sign of the real part has remained the same. On the Argand diagram this means a reflection in the x-axis (real axis).

MAKING CONNECTIONS

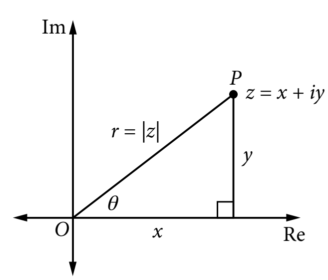
Geometric representation of conjugates

Use technology to explore the geometric representation of complex numbers and conjugates.

Modulus–argument form or polar form of a complex number

A point P on an Argand diagram may be located by Cartesian coordinates (i.e. an x-coordinate and y-coordinate, indicating horizontal displacement and vertical displacement respectively from the origin O), or alternatively by its modulus (plural ‘moduli’) and its argument:

- The **modulus** is the distance from the origin O to P.
- The **argument** is the angle at which the ray OP is inclined to the positive direction of the real axis.



Specifically, you define the modulus of z as $\bmod z = |z| = |x + iy| = \sqrt{x^2 + y^2} = r$

From this definition and the diagram, note that $x = r\cos\theta$ and $y = r\sin\theta$.

Therefore, for any complex number: $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

When a complex number is expressed in the form $r(\cos\theta + i\sin\theta)$, it is said to be in **mod–arg form** or **polar form**. The abbreviation $r\,\text{cis}\,\theta$ may be used, where $r\,\text{cis}\,\theta = r(\cos\theta + i\sin\theta)$.

The argument of $z = x + iy$ is then defined as $\arg z = \theta$ such that $x = r\cos\theta$ and $y = r\sin\theta$

Clearly, an infinite number of values of θ are possible for any complex number z, obtained by adding or subtracting multiples of 2π (or 360°). Therefore:

Define the **principal argument** to be θ such that $-\pi < \theta \leq \pi$.

Results should always be given using the principal argument.

Note: $\arg 0$ is undefined.

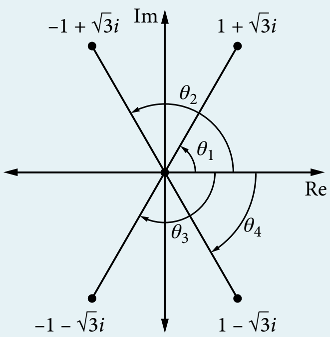
Example 8

Write each of the following in mod–arg form.

(a) $1+\sqrt{3}i$ (b) $-1+\sqrt{3}i$ (c) $-1-\sqrt{3}i$ (d) $1-\sqrt{3}i$

Solution

It is always helpful to show the complex numbers on an Argand diagram.



(a) $|z|=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$

$2\cos\theta=1$ and $2\sin\theta=\sqrt{3}$, so θ is a first-quadrant angle.

$\therefore \arg z=\theta=\frac{\pi}{3}$

$\therefore 1+\sqrt{3}i=2\operatorname{cis}\frac{\pi}{3}$

(b) $|z|=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=2$

$2\cos\theta=-1$ and $2\sin\theta=\sqrt{3}$, so θ is a second-quadrant angle.

$\therefore \arg z=\theta=\frac{2\pi}{3}$

$\therefore -1+\sqrt{3}i=2\operatorname{cis}\frac{2\pi}{3}$

(c) $|z|=\sqrt{(-1)^2+\left(-\sqrt{3}\right)^2}=2$

$2\cos\theta=-1$ and $2\sin\theta=-\sqrt{3}$, so θ is a third-quadrant angle.

$\therefore \arg z=\theta=-\frac{2\pi}{3}$ (Note the use of the principal argument.)

$\therefore -1-\sqrt{3}i=2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

(d) $|z|=\sqrt{1^2+\left(-\sqrt{3}\right)^2}=2$

$2\cos\theta=1$ and $2\sin\theta=-\sqrt{3}$, so θ is a fourth-quadrant angle.

$\therefore \arg z=\theta=-\frac{\pi}{3}$ (Again, note the use of the principal argument.)

$\therefore 1-\sqrt{3}i=2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

The result $z\times\bar{z}=|z|^2$

This useful result can be proved as follows. Let $z=x+iy$ so that $\bar{z}=x-iy$.

$\therefore$ LHS $=(x+iy)(x-iy)=x^2+y^2=$ RHS

Products in mod–arg form

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$.

Then
$$\begin{aligned} z_1\times z_2 &= r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\ &= r_1r_2(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2+i\cos\theta_1\sin\theta_2)\\ &= r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\\ &= r_1r_2\operatorname{cis}(\theta_1+\theta_2)\end{aligned}$$

This is a complex number in mod–arg form with modulus r_1r_2 and argument $(\theta_1 + \theta_2)$.

$$\therefore |z_1z_2| = |z_1| \times |z_2|$$

Also, note that $\arg z_1 + \arg z_2$ is one value of $\arg(z_1z_2)$, but not necessarily the principal value. (You may have to add or subtract a multiple of 2π to obtain the principal argument.)

$\arg(z_1z_2) = \arg z_1 + \arg z_2$ expressed in terms of the principal values

The modulus of a product is the product of the moduli.
The argument of a product is the sum of the arguments.

Example 9

If $z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, find $z_1 \times z_2$ in mod–arg form and in Cartesian form.

Hence find the exact value of $\cos\frac{5\pi}{12}$.

Solution

$$\begin{aligned} z_1 \times z_2 &= 2\sqrt{2}\left(\cos\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right) + i\sin\left(\frac{2\pi}{3}+\frac{3\pi}{4}\right)\right) \\ &= 2\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right) \quad \text{(which is in mod–arg form, but not using the principal argument)} \\ &= 2\sqrt{2}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right) \quad \text{(subtracting } 2\pi \text{ to find the principal arg)} \end{aligned}$$

To find $z_1 \times z_2$ in Cartesian form:

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}i$$

$$z_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$$

$$\therefore z_1 \times z_2 = (-1+\sqrt{3}i)(-1+i) = (1-\sqrt{3}) + (-1-\sqrt{3})i$$

So $2\sqrt{2}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right) = (1-\sqrt{3}) + (-1-\sqrt{3})i$ in Cartesian form.

Equating the real parts (because you are trying to prove a result involving $\cos\frac{5\pi}{12}$):

$$2\sqrt{2}\cos\frac{-7\pi}{12} = 1-\sqrt{3} \qquad \therefore \cos\frac{-7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

But $\cos\frac{-7\pi}{12} = \cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$ (as $\cos x$ is an even function and $\cos(\pi - \theta) = -\cos\theta$)

$$\therefore -\cos\frac{5\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}} \quad \text{and so} \quad \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

EXPLORING FURTHER

Polar form of a complex number

Use technology to explore the polar form of a complex number.

Quotients in mod–arg form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned}
\text{Then } \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\
&= \frac{r_1\left((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\right)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)} \\
&= \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right) \\
&= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)
\end{aligned}$$

This is a complex number in mod–arg form with modulus $\frac{r_1}{r_2}$ and argument $(\theta_1 - \theta_2)$.

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Also, note that $\arg z_1 - \arg z_2$ is one value of $\arg\frac{z_1}{z_2}$, but not necessarily the principal value. (You may have to add or subtract a multiple of 2π to find the principal argument.)

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2, z_2 \neq 0,$ expressed in terms of the principal values.

It follows from these results that $\left|\frac{1}{z}\right| = \frac{|1|}{|z|} = \frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right) = \arg 1 - \arg z = -\arg z$.

The modulus of a quotient is the quotient of the moduli.
The argument of a quotient is the difference of the arguments.

Example 10

If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, find $\frac{z_1}{z_2}$ in mod–arg form.

Solution

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \text{and} \quad z_2 = 2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$$

$$\therefore \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{4} - \frac{-\pi}{6}\right) = \frac{\sqrt{2}}{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Two special results

1 If $z = r(\cos\theta + i\sin\theta)$ then the conjugate $\bar{z} = r\operatorname{cis}(-\theta)$

2 If $z = r(\cos\theta + i\sin\theta)$ then $\frac{1}{z} = \frac{1}{r}\operatorname{cis}(-\theta)$

This second result can be proved as follows:

Method 1

$$\frac{1}{z} = \frac{1\operatorname{cis}0}{r\operatorname{cis}\theta} = \frac{1}{r}\operatorname{cis}(0 - \theta) = \frac{1}{r}\operatorname{cis}(-\theta)$$

Method 2

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{r\operatorname{cis}(-\theta)}{r^2} = \frac{1}{r}\operatorname{cis}(-\theta)$$

Powers using mod–arg form

De Moivre's theorem (named for the French mathematician Abraham de Moivre (1667–1754)) states:

> If $z = r(\cos\theta + i\sin\theta)$ and n is an integer,
> then $z^n = r^n(\cos n\theta + i\sin n\theta)$
> $|z^n| = |z|^n$ and $\arg z^n = n\arg z$

It follows from these results that $\left|\dfrac{1}{z^n}\right| = \dfrac{|1|}{|z^n|} = \dfrac{1}{|z|^n}$ and $\arg\left(\dfrac{1}{z^n}\right) = \arg 1 - \arg z^n = -n\arg z$.

You will prove this theorem in question **13** of Exercise 1.2 below.

Example 11

If $z = 1 + i$, express z^{-10} in Cartesian form $(x + iy)$.

Solution

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\begin{aligned}\therefore\quad z^{-10} &= \left(\sqrt{2}\right)^{-10}\left(\cos\left(\frac{-10\pi}{4}\right) + i\sin\left(\frac{-10\pi}{4}\right)\right)\\ &= \frac{1}{32}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)\\ &= -\frac{i}{32}\end{aligned}$$

Example 12

Let $z_1 = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$, $z_2 = \sqrt{2}\left(\cos\dfrac{-\pi}{4} + i\sin\dfrac{-\pi}{4}\right)$.

(a) Find the smallest positive integer n for which $\left(\dfrac{z_1}{z_2}\right)^n$ is a real number.

(b) If $z = \dfrac{z_1^3}{z_2^5}$, find z in Cartesian form.

Solution

(a) $\arg z_1 = \dfrac{\pi}{3}$ and $\arg z_2 = -\dfrac{\pi}{4}$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3} - \frac{-\pi}{4} = \frac{7\pi}{12}$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right)^n = \frac{7n\pi}{12}$$

Now $\left(\dfrac{z_1}{z_2}\right)^n$ is a real number when $\arg\left(\dfrac{z_1}{z_2}\right)^n$ is an integer multiple of π, because that makes the argument zero (so the imaginary part is zero). $n = 12$ is the smallest positive value of n that makes this happen.

(b)
$$\begin{aligned} z &= \frac{z_1^3}{z_2^5}\\ &= \frac{2^3(\cos\pi + i\sin\pi)}{\left(\sqrt{2}\right)^5\left(\cos\frac{-5\pi}{4} + i\sin\frac{-5\pi}{4}\right)}\\ &= \sqrt{2}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)\\ &= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\\ &= 1 + i\end{aligned}$$

Basic identities

- $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2,\ z_2 \neq 0$
- $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$
- $\left|\dfrac{1}{z^n}\right| = \dfrac{1}{|z^n|}$ and $\arg\left(\dfrac{1}{z^n}\right) = -n\arg z,\ z \neq 0$
- $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$
- $\overline{z}_1\overline{z}_2 = \overline{z_1 z_2}$
- $z\overline{z} = |z|^2$
- $z + \overline{z} = 2\operatorname{Re}(z)$
- $z - \overline{z} = 2i\operatorname{Im}(z)$

EXERCISE 1.2 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A POINT

1 If $z = 2 + i$ and $w = -3 - 4i$, represent each of the following on the complex plane.

(a) z (b) $\overline{z}$ (c) $z\overline{z}$ (d) $3z$ (e) $-2z$ (f) $\dfrac{1}{z}$ (g) $z + w$
(h) $-w$ (i) $z - w$ (j) z^2 (k) $\operatorname{Re}(z)$ (l) $\operatorname{Im}(z)$

2 If $z = 2\left(\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3}\right)$, then $z^4 = \ldots$

A $16\left(\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3}\right)$ B $16\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

C $16\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ D $16\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

3 If $z = \overline{z}$, then $\arg z = \ldots$

A π B $\dfrac{\pi}{2}$ C 0 D 0 or π

4 Express each of the following in mod–arg form. (Give the argument in radians and in exact form.)

(a) $2 - 2i$ (b) $-\sqrt{3} + i$ (c) $-6 - 6i$ (d) $4i$ (e) -4
(f) $-3 - \sqrt{3}i$ (g) $2\sqrt{3} - 2i$ (h) $\sqrt{2} + \sqrt{2}i$

5 Convert each of the following into Cartesian form.

(a) $4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ (b) $8\left(\cos\dfrac{-\pi}{4} + i\sin\dfrac{-\pi}{4}\right)$

(c) $6\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$ (d) $2\left(\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3}\right)$

6 For each of the following, find both zw and $\dfrac{z}{w}$ in mod–arg form.

(a) $z = 4\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right),\ w = 4\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ (b) $z = 5\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right),\ w = 3\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

(c) $z = \sqrt{2}\left(\cos\dfrac{-3\pi}{4} + i\sin\dfrac{-3\pi}{4}\right),\ w = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

7 If $z = x + iy$, prove the following.

(a) $|z| = |\overline{z}|$ (b) $z\overline{z} = |z|^2$ (c) $z + \dfrac{|z|^2}{z} = 2\operatorname{Re}(z)$

8 On an Argand diagram, mark points A, B and C to represent complex numbers z, w and $z + w$. Give a geometrical explanation to show that $|z + w| \leq |z| + |w|$.

9 Find the following in Cartesian form.

(a) $\left[2\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)\right]^5$ (b) $\left[\sqrt{2}\left(\cos\frac{-3\pi}{4}+i\sin\frac{-3\pi}{4}\right)\right]^8$ (c) $(\sqrt{3}+i)^6$

(d) $(1-i)^5$ (e) $(\sqrt{3}-i)^4$ (f) $\frac{1}{(2\sqrt{3}+2i)^5}$ (g) $(-4-4\sqrt{3}i)^{-3}$

(h) $(1-i)^3(2+2i)^4$ (i) $\frac{(1+i)^3}{(1-i)^4}$ (j) $\frac{(\sqrt{3}+i)^6}{(1-i)^8}$

10 If $z=3-4i=5(\cos\theta+i\sin\theta)$, find the following in $x+iy$ form.

(a) $25(\cos 2\theta+i\sin 2\theta)$ (b) $5(\sin\theta-i\cos\theta)$ (c) $\frac{1}{5}(\cos\theta-i\sin\theta)$

11 If $z=r(\cos\theta+i\sin\theta)$, show that $\frac{z}{z^2+r^2}$ is real.

12 Let $z=\sqrt{3}+i$ and $w=z\times(\cos\theta+i\sin\theta)$ where $-\pi<\theta\le\pi$.

(a) Find the value of θ if w is purely imaginary and $\text{Im}(w)>0$.

(b) Find the value of $\arg(z+w)$.

13 (a) If $z=\cos\theta+i\sin\theta$, prove by induction that $z^n=\cos n\theta+i\sin n\theta$ for all positive integers n. (This is the proof of de Moivre's theorem for positive integers.)

(b) By writing $z^{-n}=\frac{1}{z^n}$, complete the proof of de Moivre's theorem for negative integers.

14 Use de Moivre's theorem to prove that the conjugate of a power is equal to the power of the conjugate, i.e. let $z=r(\cos\theta+i\sin\theta)$ and prove that $\overline{z^n}=(\overline{z})^n$.

15 We have already proved (earlier and in question **14**) that:

- $z+\overline{z}=2\text{Re}(z)$ and $z-\overline{z}=2\text{Im}(z)\times i$
- the conjugate of a sum is equal to the sum of the conjugates
- the conjugate of a difference is equal to the difference of the conjugates
- the conjugate of a product is equal to the product of the conjugates
- the conjugate of a quotient is equal to the quotient of the conjugates
- the conjugate of a power is equal to the power of the conjugate.
- It is also obvious that the conjugate of a real number is itself, i.e. if $z=x+0i$ then $\overline{z}=x-0i=z$.

Use these properties of conjugates to answer the following.

(a) Show that $z^n+(\overline{z})^n=2\text{Re}(z^n)$.

(b) Simplify $(1+\sqrt{3}i)^{10}+(1-\sqrt{3}i)^{10}$.

16 Consider the cubic polynomial $P(x)=ax^3+bx^2+cx+d$ for which all the coefficients a, b, c and d are real. Let the complex number z be a root of the equation $P(x)=0$. Show that $\overline{z}$ is also a root of $P(x)=0$.

1.3 OTHER REPRESENTATIONS OF COMPLEX NUMBERS

Euler's formula

A very useful result is Euler's formula, which states that $e^{ix}=\cos x+i\sin x$, for real x.

It may seem strange to have an exponential function as the sum of two trigonometric functions, so the result needs to be proved.

Proof

Let $f(x)=\cos x+i\sin x$ [1]

Differentiate with respect to x:

$$\begin{aligned} f'(x) &= -\sin x+i\cos x \\ &= i(\cos x+i\sin x) \\ &= i\,f(x) \end{aligned}$$

Hence $\dfrac{f'(x)}{f(x)} = i$.

Integrate both sides with respect to x:

$$\int \frac{f'(x)}{f(x)}\,dx = i\int dx$$

Hence $\log_e |f(x)| = ix + C$ [2]

Using [1] you have: $f(0) = \cos 0 + i\sin 0 = 1$

Substitute into [2]: $\log_e 1 = C$ so $C = 0$

[2] becomes: $\log_e |f(x)| = ix$

So $f(x) = e^{ix}$

Hence $e^{ix} = \cos x + i\sin x$

Thus $e^{-ix} = \cos(-x) + i\sin(-x)$

$= \cos x - i\sin x$

In general, when $|z| \neq 1, z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Example 13

Write each complex number in both polar and Cartesian form.

(a) $e^{\frac{i\pi}{6}}$ **(b)** $e^{\frac{-i\pi}{3}}$ **(c)** $e^{\frac{3\pi i}{4}}$ **(d)** $e^{\frac{-5\pi i}{6}}$ **(e)** $e^{-i\pi}$ **(f)** $e^{1+\frac{i\pi}{6}}$

Solution

(a) $e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

(b) $e^{\frac{-i\pi}{3}} = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) $e^{\frac{3\pi i}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}(-1+i)$

(d) $e^{\frac{-5\pi i}{6}} = \cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right) = -\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(e) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -\cos 0 + i\sin 0 = -1$

(f) $e^{1+\frac{i\pi}{6}} = e \times e^{\frac{i\pi}{6}} = e\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{e\sqrt{3}}{2} + \frac{e}{2}i$

Example 14

Write each complex number in the form $re^{i\theta}$, giving any decimal answers correct to two decimal places.

(a) $3(\cos 2 + i\sin 2)$ **(b)** $-1 + i\sqrt{3}$ **(c)** $2 + 3i$ **(d)** $2(\cos 1.5 - i\sin 1.5)$ **(e)** $-3 - 3i$

Solution

(a) $3(\cos 2 + i\sin 2) = 3e^{2i}$

(b) $-1 + i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{\frac{2\pi i}{3}}$

(c) $2+3i=\sqrt{13}\left(\frac{2}{\sqrt{13}}+\frac{3}{\sqrt{13}}i\right)=\sqrt{13}(\cos\theta+i\sin\theta)$ where $\theta=\tan^{-1}\left(\frac{3}{2}\right)\approx 0.98$

$=\sqrt{13}e^{0.98i}$

(d) $2(\cos 1.5-i\sin 1.5)=2(\cos(-1.5)+i\sin(-1.5))=2e^{-1.5i}$

(e) $-3-3i=3(-1-i)=3\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=3\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right)=3\sqrt{2}e^{\frac{-3\pi i}{4}}$

Powers of complex numbers

Since any complex number can be written in the exponential form, $e^{i\theta}$, it is easy to find powers of this number as $(e^{i\theta})^n=e^{in\theta}$.

Example 15

(a) Write $z=1+i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 **(ii)** z^3 **(iii)** z^4 **(iv)** $\sqrt{z}$ **(v)** z^{-1}

Solution

(a) $z=1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}e^{\frac{i\pi}{4}}$

(b) **(i)** $z^2=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^2=2e^{\frac{i\pi}{2}}=2(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})=2i$

(ii) $z^3=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^3=2\sqrt{2}e^{\frac{3\pi i}{4}}=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=-2+2i$

This answer could also have been obtained using $z^3=z^2\times z=2i(1+i)=-2+2i$.

(iii) $z^4=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^4=4e^{i\pi}=4(\cos\pi+i\sin\pi)=-4$

(iv) $\sqrt{z}=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{\frac{1}{2}}=\sqrt[4]{2}e^{\frac{i\pi}{8}}=\sqrt[4]{2}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)=\sqrt[4]{2}(0.9239+0.3827i)=1.099+0.4204i$

(v) $z^{-1}=\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{-1}=\frac{1}{\sqrt{2}}e^{\frac{-i\pi}{4}}=\frac{1}{\sqrt{2}}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\frac{1}{2}-\frac{1}{2}i$

Example 16

Given $z_1=2e^{\frac{i\pi}{6}}$, $z_2=3e^{\frac{-i\pi}{3}}$ and $z_3=e^{\frac{3i\pi}{4}}$, find the polar form for each of the following.

(a) $z_1\times z_2$ **(b)** $z_2\times z_3$ **(c)** $z_1{}^2\times z_2$ **(d)** $\frac{z_1}{z_2}$

(e) $\frac{z_2}{z_3}$ **(f)** $\frac{z_1{}^2\times z_2}{z_3}$ **(g)** On the Argand diagram, plot z_1, z_2 and $z_1\times z_2$.

(h) On the Argand diagram, plot z_2, z_3 and $\frac{z_2}{z_3}$.

Solution

(a) $z_1 \times z_2 = 2e^{\frac{i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 6e^{\frac{i\pi}{6}+\frac{-i\pi}{3}} = 6e^{\frac{-i\pi}{6}}$

(b) $z_2 \times z_3 = 3e^{\frac{-i\pi}{3}} \times e^{\frac{3i\pi}{4}} = 3e^{\frac{-i\pi}{3}+\frac{3i\pi}{4}} = 3e^{\frac{5i\pi}{12}}$

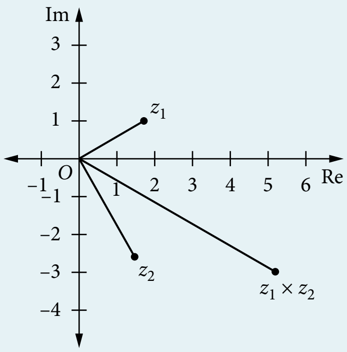
(c) $z_1{}^2 \times z_2 = 2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 12e^{\frac{i\pi}{3}+\frac{-i\pi}{3}} = 12e^0 \,(=12)$

(d) $\dfrac{z_1}{z_2} = \dfrac{2e^{\frac{i\pi}{6}}}{3e^{\frac{-i\pi}{3}}} = \dfrac{2}{3}e^{\frac{i\pi}{6}-\frac{-i\pi}{3}} = \dfrac{2}{3}e^{\frac{i\pi}{2}} \left(=\dfrac{2}{3}i\right)$

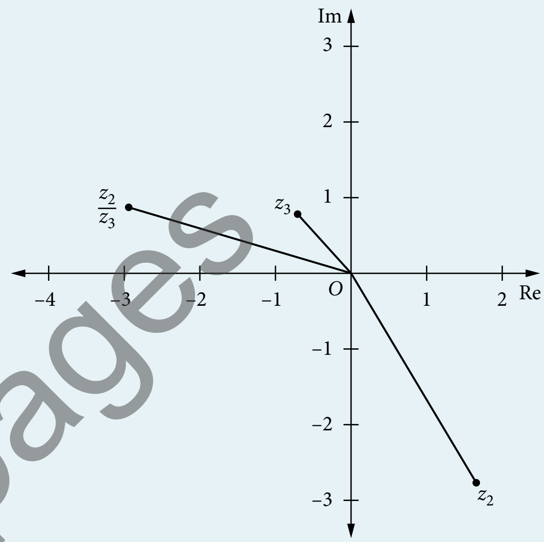
(e) $\dfrac{z_2}{z_3} = \dfrac{3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = 3e^{\frac{-i\pi}{3}-\frac{3i\pi}{4}} = 3e^{\frac{-13i\pi}{12}} = 3e^{\frac{11i\pi}{12}}$

(f) $\dfrac{z_1{}^2 \times z_2}{z_3} = \dfrac{2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = \dfrac{12e^0}{e^{\frac{3i\pi}{4}}} = 12e^{\frac{-3i\pi}{4}}$

(g)



(h)



Geometrical representation of products involving complex numbers—consolidation and summary

Multiplication of a complex number z by a real number k:

- $\arg(kz) = \arg k + \arg z$
  If k is a positive real number, then $\arg k = 0$, so $\arg(kz) = \arg z$.
  If k is a negative real number, then $\arg k = \pi$, so $\arg(kz) = \pi + \arg z = \pi + \arg z - 2\pi = -(\pi - \arg z)$.
  (Note that 2π is subtracted to find the principal argument.)
- $|kz| = |k| \times |z|$, i.e. there is a scaling by a factor of $|k|$
  If k is a negative real number, then the direction from the origin O to the point representing kz is opposite to the direction from O to the point representing z.

Multiplication of a complex number z by i:

- $\arg(iz) = \arg i + \arg z = \dfrac{\pi}{2} + \arg z$
- $|iz| = |i| \times |z| = |z|$ as $|i| = 1$
- Hence multiplication by i causes an anticlockwise rotation by $\dfrac{\pi}{2}$ about the origin O, with no change to the modulus.

Multiplication of a complex number z by ki, where k is a real number:

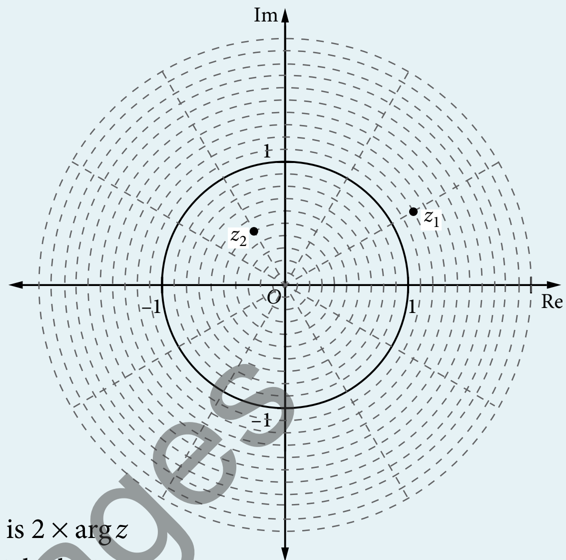
- This combines the two cases above.
- Rotate by $\dfrac{\pi}{2}$ anticlockwise about O and then scale by a factor of $|k|$, remembering also to reverse the direction if k is negative.

Multiplication of a complex number z by another complex number $r(\cos\theta + i\sin\theta)$:

- $\arg(z \times r\operatorname{cis}\theta) = \arg z + \arg(r\operatorname{cis}\theta) = \arg z + \theta$
- $|z \times r\operatorname{cis}\theta| = |z| \times |r\operatorname{cis}\theta| = |z| \times r$
- To multiply by $r\operatorname{cis}\theta$, rotate by θ anticlockwise about O and then scale by a factor of r.

Example 17

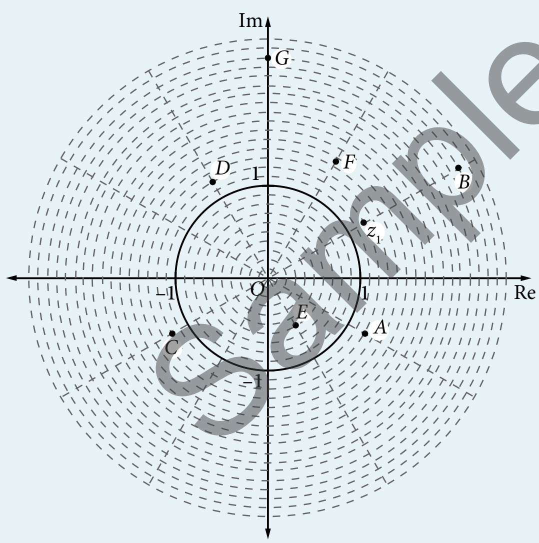
The Argand diagram at right shows the unit circle as well as points representing the complex numbers z_1 and z_2.
For **(a)** $z = z_1$ and **(b)** $z = z_2$, mark points A, B, C, D, E, F, G to represent $\overline{z}$, $2z$, $-z$, iz, $-\frac{1}{2}iz$, z^2 and $(1+\sqrt{3}i)z$.



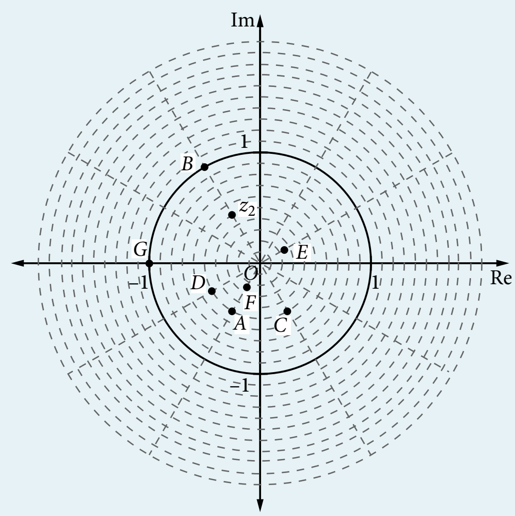
Solution

A: $\overline{z}$ is the reflection of z in the real axis
B: $2z$ is z scaled by a factor of 2
C: $-z$ is z scaled by a factor of -1
(i.e. reflected back through O)
D: iz is z rotated by $\frac{\pi}{2}$ anticlockwise about O
E: $-\frac{1}{2}iz$ is iz scaled by a factor of $-\frac{1}{2}$
F: z^2 has a modulus that is $(\operatorname{mod} z)^2$ and an argument that is $2 \times \arg z$
G: $1+\sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$, so $(1+\sqrt{3}i)z$ is found by rotating anticlockwise by $\frac{\pi}{3}$ and then doubling the modulus.

(a)



(b)



Example 18

Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively.

(a) Find the complex number represented by B.
(b) The square is now rotated anticlockwise 45° about O to form $OA'B'C'$. Find the complex numbers represented by A', B' and C'.
(c) E is the point of intersection of the diagonals of the square $OA'B'C'$. What complex number does E represent?