PEARSON

SPECIALIST MATHEMATICS

QUEENSLAND

STUDENT BOOK

11

UNITS 1 & 2

QCE 2019 SYLLABUS

Contents

UNIT 2

Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non CAS)
- CASIO fx-CG50AU

Throughout the student book are Technology worked examples strategically placed within the theory. These are suitable for both the TI-Nspire CX (non CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

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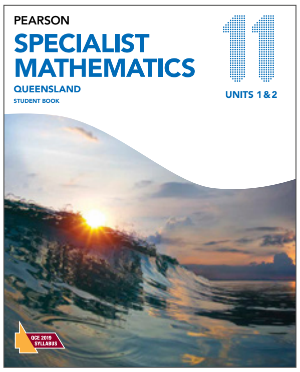
SPECIALIST MATHEMATICS 11

QUEENSLAND

UNITS 1 & 2



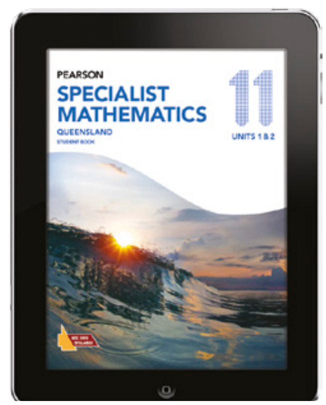
Specialist Mathematics 11
Student book

Student book

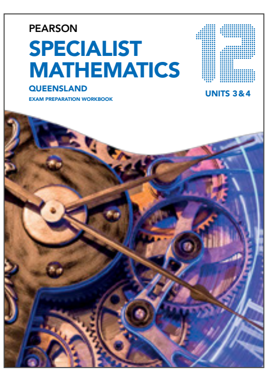
The student book has been authored by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



Specialist Mathematics 11
eBook



Specialist Mathematics 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Specialist Mathematics 11 Queensland Units 1 & 2

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

Explore further

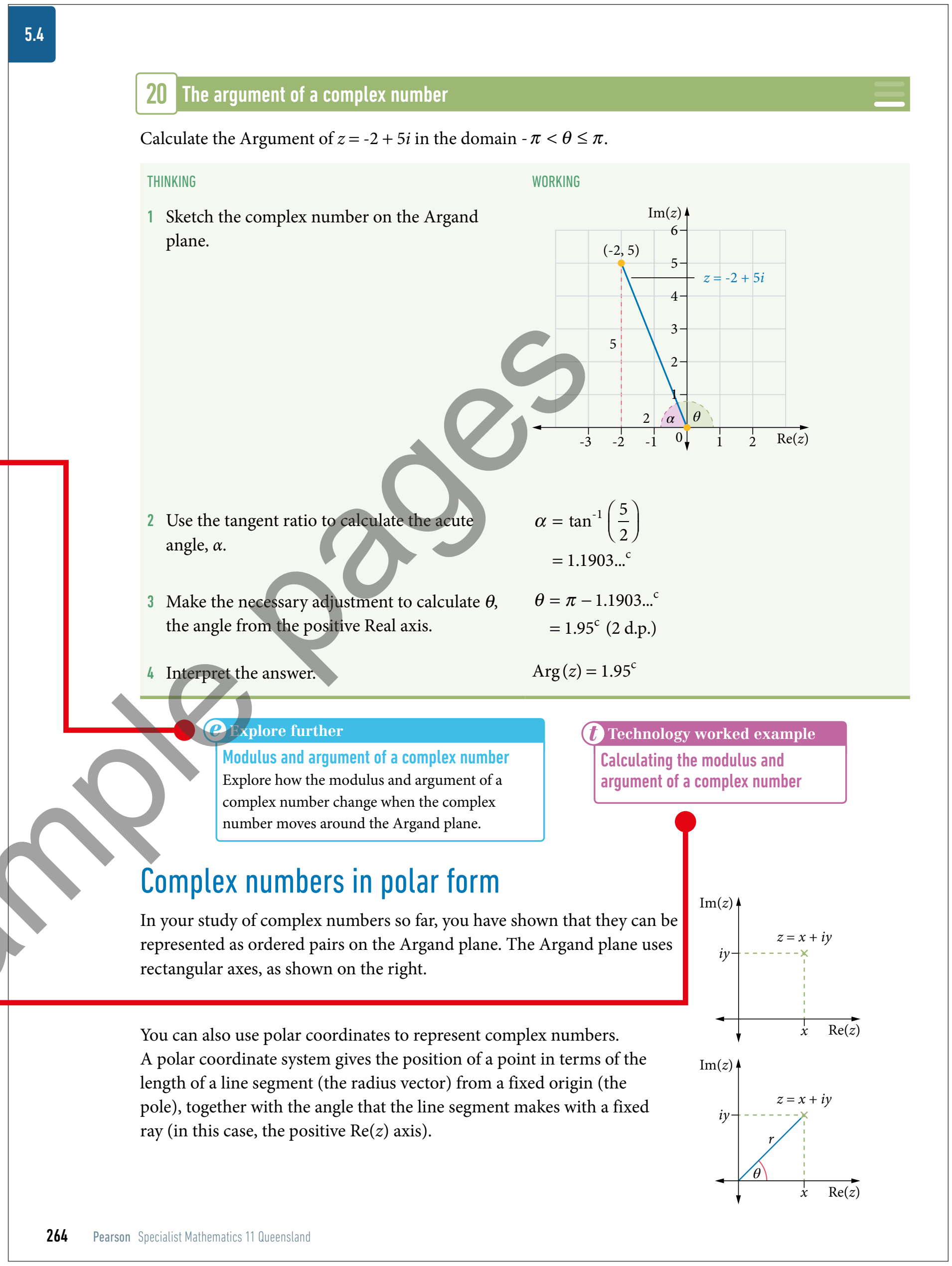
This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Technology worked examples

These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.


5.4

20 The argument of a complex number

Calculate the Argument of $z = -2 + 5i$ in the domain $-\pi < \theta \leq \pi$.

THINKING	WORKING
1 Sketch the complex number on the Argand plane.	
2 Use the tangent ratio to calculate the acute angle, α.	$\alpha = \tan^{-1}\left(\frac{5}{2}\right)$ $= 1.1903...^c$
3 Make the necessary adjustment to calculate θ, the angle from the positive Real axis.	$\theta = \pi - 1.1903...^c$ $= 1.95^c$ (2 d.p.)
4 Interpret the answer.	$\text{Arg}(z) = 1.95^c$

Explore further
Modulus and argument of a complex number
Explore how the modulus and argument of a complex number change when the complex number moves around the Argand plane.

Technology worked example
Calculating the modulus and argument of a complex number

Complex numbers in polar form

In your study of complex numbers so far, you have shown that they can be represented as ordered pairs on the Argand plane. The Argand plane uses rectangular axes, as shown on the right.

You can also use polar coordinates to represent complex numbers. A polar coordinate system gives the position of a point in terms of the length of a line segment (the radius vector) from a fixed origin (the pole), together with the angle that the line segment makes with a fixed ray (in this case, the positive Re(z) axis).

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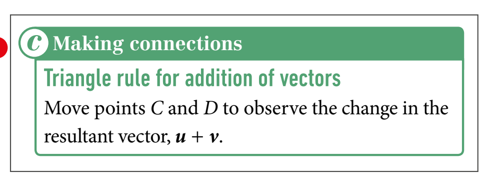

Making connections
Triangle rule for addition of vectors
Move points C and D to observe the change in the resultant vector, $\boldsymbol{u} + \boldsymbol{v}$.

Key information

Key information and rules are highlighted throughout the chapter.

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

Meeting the needs of the QCE syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Tech-free questions

These questions are designed to provide students with the opportunity to practice algebraic manipulations to prepare them for technology-free examination papers.

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.

This same logic works in general: The value of ${}^nP_r$ can be obtained by dividing $n!$ by $(n-r)!$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

13 Using the factorial form of the permutation formula

Calculate ${}^8P_3$ using factorial notation and interpret the result.

THINKING	WORKING
1 Substitute the values of n and r into the formula: ${}^nP_r = \frac{n!}{(n-r)!}$	For ${}^8P_3$, $n = 8$ and $r = 3$: ${}^8P_3 = \frac{8!}{(8-3)!}$
2 Express as a fraction in factorial form.	$= \frac{8!}{5!}$
3 Simplify the expression by cancelling the common factors.	$= \frac{8\times7\times6\times\cancel{5!}}{\cancel{5!}}$
4 Calculate the expression.	$= 8\times7\times6$ $= 336$
5 Interpret the result.	There are 336 arrangements of 3 objects selected from a group of 8.

It was demonstrated earlier that the number of ways to arrange n objects in a row is equal to $n!$. In other words, ${}^nP_n = n!$. Applying the factorial form of the permutation formula to the case of ${}^nP_n$ gives

$$\begin{aligned}{}^nP_n &= n!\\ &= \frac{n!}{(n-n)!}\\ &= \frac{n!}{0!}\end{aligned}$$

Since ${}^nP_n = n!$, it must be the case that $0!$ is equal to 1 if the factorial form of the permutation formula is to remain valid in this particular situation.

While 0! arises infrequently in practice, it is useful to remember that it is defined to have the value 1.

$0! = 1$

9 (a) Which of the following is equivalent to $(3n)!$?

A $3n!$ B $3!n!$ C $3n(n-1)!$ D $3n(3n-1)!$

(b) Explain the common errors in the three incorrect alternatives.

WARNING

The inverse function of $\sin(\theta)$ is $\sin^{-1}(\theta)$, but note that $\sin^{-1}(\theta) \neq \frac{1}{\sin(\theta)}$, i.e. it is not the reciprocal function. This applies to all the inverse trigonometric functions.

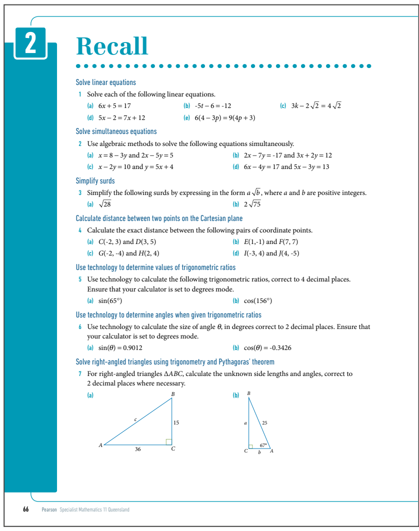
For example, $\cos^{-1}(-1) = \pi$, because $\cos(\pi) = -1$.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

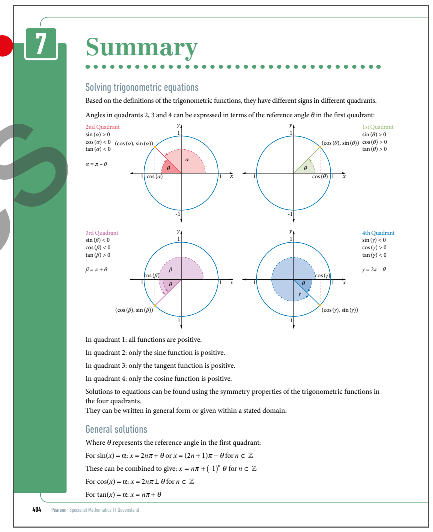


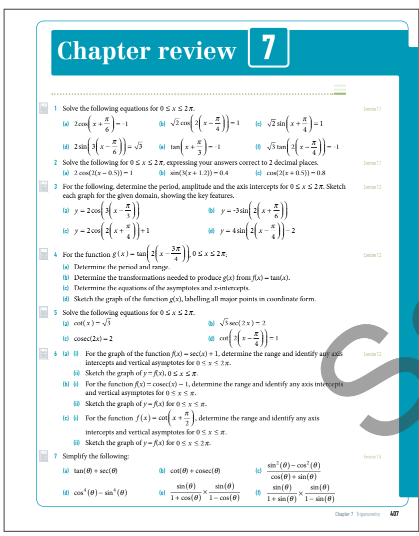
Recall

Each chapter begins with a review of assumed knowledge for the chapter.

Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.



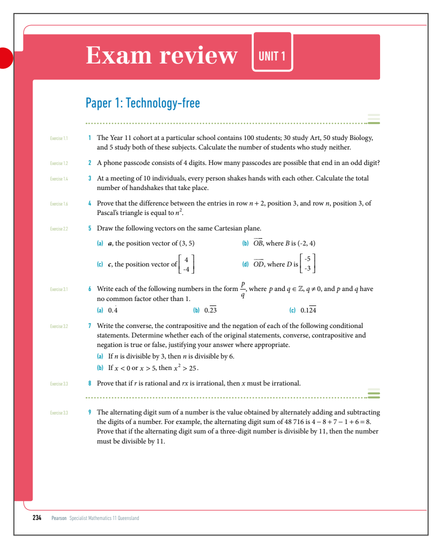


Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.

Exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each Unit.



3

Introduction to proof

3

Recall

Identify numbers with certain properties

1 Identify the numbers from 0, 1, 2, 3, …, 10 with the following properties.

(a) even (b) odd (c) prime

(d) divisible by 4 (e) factor of 4 (f) perfect square

Express a given integer as a product of prime powers

2 Express each of the following numbers as a product of prime powers.

(a) 8 (b) 12 (c) 90

(d) 63 (e) 50

Simplify surds

3 Simplify each of the following.

(a) $\sqrt{50}$ (b) $2\sqrt{28} - 3\sqrt{63} + 8\sqrt{7}$ (c) $(5 - 2\sqrt{2})(5 + 2\sqrt{2})$

Rationalise denominators of fractions involving surds

4 Express each of the following fractions with integer denominators.

(a) $\frac{5}{\sqrt{6}}$ (b) $\frac{2}{3 - 2\sqrt{5}}$ (c) $\frac{4 + \sqrt{3}}{4 - \sqrt{3}}$

Simplify algebraic expressions involving surds

5 Simplify each of the following. Assume x and y are positive real numbers.

(a) $\sqrt{x^6}$ (b) $\sqrt{y^7}$ (c) $\left(\sqrt{x} + \sqrt{y}\right)^2$

Simplify algebraic expressions involving indices

6 Simplify each of the following.

(a) $3^{n+1} \times 3^{n-1}$ (b) $81 \times 3^{n-5}$ (c) $\frac{3^{n+5}}{81}$

Solve simple inequalities

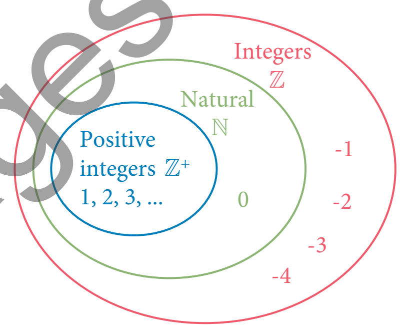
7 Solve the following inequalities.

(a) $3x + 5 \leq 17$ (b) $-2x + 1 > 7$

Rational and irrational numbers 3.1

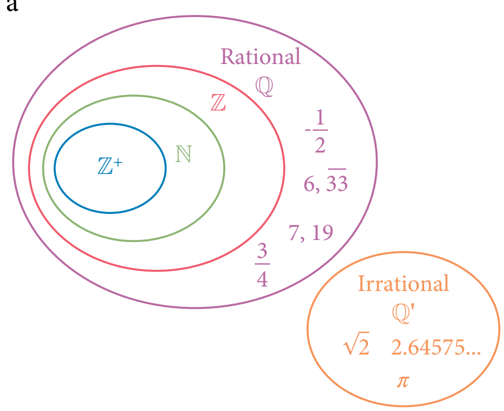
Early civilisations used numbers exclusively to count things; there was no need for negative numbers, fractional numbers or even the number zero. These other types of numbers were introduced gradually over many thousands of years with contributions from many cultures. Today, much of mathematics is based on a foundation of numbers, and it is therefore important to understand the different types of numbers that mathematicians work with. This chapter focuses on real numbers; that is, numbers that can be placed on a number line. (In Chapter 5, you will be introduced to the concept of imaginary numbers, which cannot be placed on a number line.)

The most fundamental type of number is a *positive integer*. The symbol $\mathbb{N}$ (or less commonly, N or J) is used to denote the set of non-negative integers, {0, 1, 2, 3...}; non-negative integers are also known as *natural numbers*. Note that some sources do not include 0 in the set of natural numbers, and use the terms 'natural number' and 'positive integer' interchangeably.



The set of integers, $\mathbb{Z}$ (or less commonly, Z), includes all *whole numbers* (both positive and negative), and zero (which is neither positive nor negative). The symbol $\mathbb{Z}^+$ is used to denote just the set of positive integers, {1, 2, 3...}, while the symbol $\mathbb{Z}^-$ is used to denote just the set of negative integers, {-1, -2, -3...}.

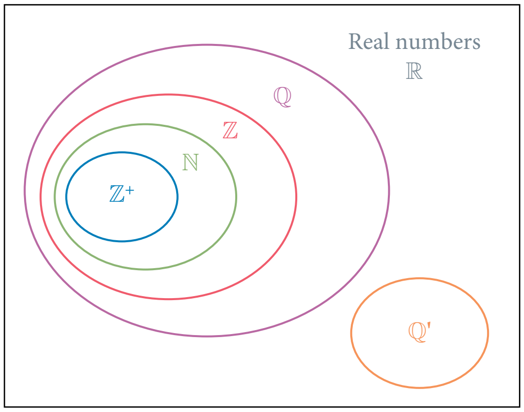
A *rational number* is any number that can be expressed in the form $\frac{p}{q}$ where p and q are both integers, and $q \neq 0$. Any number with a decimal representation that either terminates or repeats in a regular pattern (recurs) is rational. The symbol $\mathbb{Q}$ (or less commonly, Q) is used to denote the set of all rational numbers. Note that all integers are rational (since an integer, n, can be expressed as $\frac{n}{1}$).

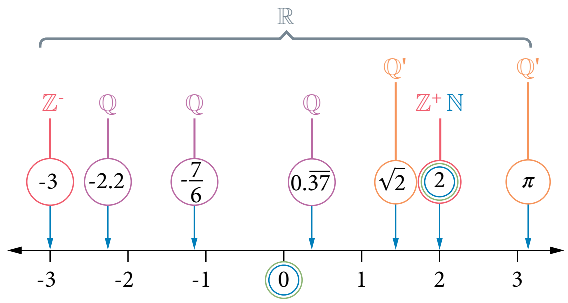


Numbers that are not rational, but can still be placed on a number line, are said to be *irrational*. The symbol $\mathbb{Q}'$ (or less commonly, Q or I) is used to denote the set of irrational numbers. The number π is known to be irrational, however, it is often approximated as $\frac{22}{7}$. The decimal representations of irrational numbers go on forever in no predictable pattern; thus, irrational number cannot be written exactly using decimals.

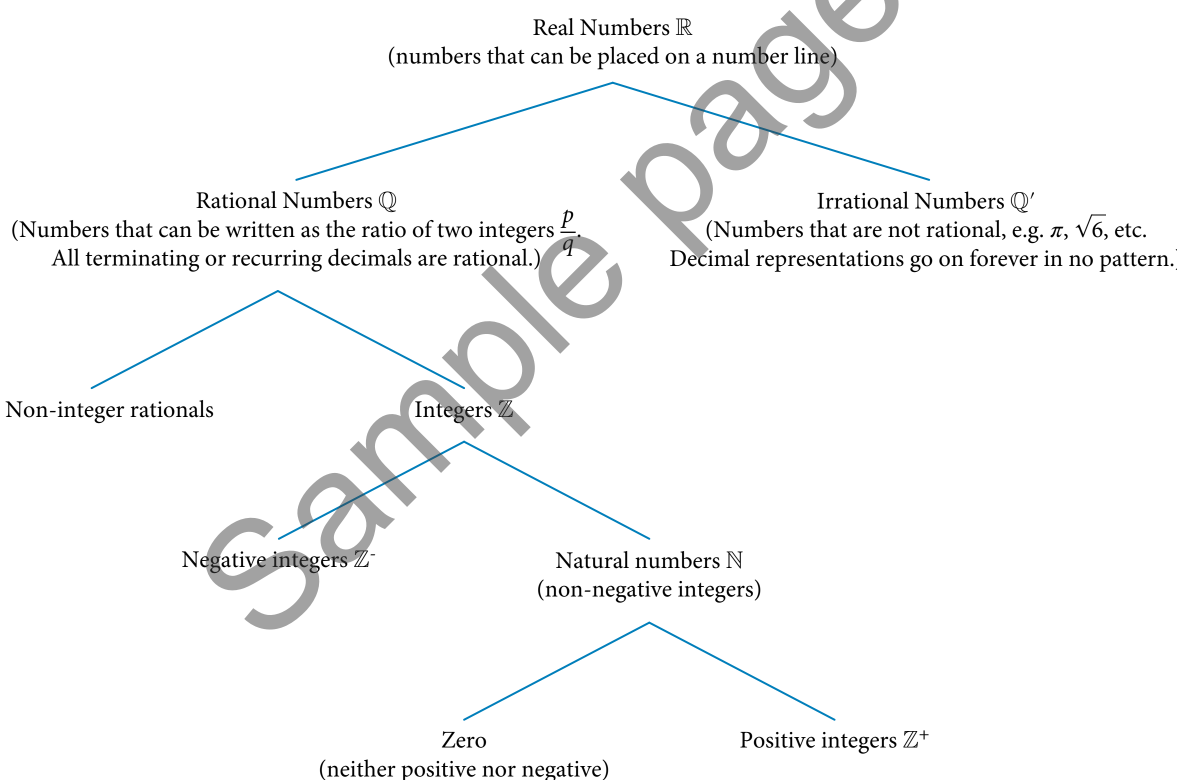
As well as the number π, any number that can only be expressed exactly using a radical symbol $\left(\sqrt{\ }\right)$ is *irrational*. For example, $\sqrt{2}$ is irrational. Note that $\sqrt{9}$ is not irrational because it is possible to express this number without the radical symbol $\left(\sqrt{9} = 3\right)$.

The sets of rational and irrational numbers together comprise the set of real numbers, denoted by the symbol $\mathbb{R}$ (or less commonly, R). As stated previously, the set of real numbers can be thought of as the set of all numbers that can be placed on a number line; see diagram below.





The following diagram summarises the symbols and relationships between the different sets of numbers described.



1 Classifying numbers

For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) -7

THINKING	WORKING
1 Consider whether or not the number belongs to each of the sets listed.	-7 is a negative integer. Therefore it is also an integer, a rational number, and a real number.
2 Answer the question.	Hence, -7 is an element of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}^-$.

(b) $4.\overline{23}$

1 Consider whether or not the number belongs to each of the sets listed.	$4.\overline{23}$ is a recurring decimal. Therefore, it is a non-integer rational number.
2 Answer the question.	Hence, $4.\overline{23}$ is an element of $\mathbb{R}$ and $\mathbb{Q}$.

(c) $6\sqrt{10}$

1 Consider whether or not the number belongs to each of the sets listed. If a number cannot be expressed without a radical symbol, it is irrational. Further, the product (or sum, or difference) of an irrational number and a rational number is irrational.	The number is part of the Real number system, $\mathbb{R}$, but cannot be written without a radical. It is therefore classified as an Irrational number, $\mathbb{Q}'$.
2 Answer the question.	Hence, $6\sqrt{10}$ is an element of $\mathbb{R}$ and $\mathbb{Q}'$.

(d) $\sqrt{\frac{25}{49}}$

1 Consider whether or not the number belongs to each of the sets listed. If a number cannot be expressed without a radical symbol, it is irrational.	$\sqrt{\frac{25}{49}} = \sqrt{\left(\frac{5}{7}\right)^2} = \frac{5}{7}$ Thus, this number is rational.
2 Answer the question.	Hence, $\sqrt{\frac{25}{49}}$ is an element of $\mathbb{R}$ and $\mathbb{Q}$.

(e) $\sqrt{-100}$

1 Consider whether or not the number belongs to each of the sets listed.	Square roots of negative numbers are not real numbers (and hence, do not belong to any of the other sets listed).
2 Answer the question.	$\sqrt{-100}$ does not belong to any of the sets listed.

As previously stated, all rational numbers can be expressed as either terminating or recurring decimals. The following example illustrates how to express a rational number in decimal form without the aid of a calculator.

2 Expressing rational numbers as decimals

Express each of the following rational numbers as a terminating or recurring decimal.

(a) $\frac{3}{25}$

THINKING	WORKING
If the denominator of a fraction divides into 10, 100, 1000, etc. then the number can readily be written as a terminating decimal by first rewriting the fraction with a denominator that is a power of 10.	$\frac{3}{25} = \frac{12}{100}$ $= 0.12$

(b) $\frac{3}{80}$

If it is not obvious how to express the fraction as a decimal, use short division. It may also be helpful to first divide the top and bottom of the fraction by 10 to obtain a single-digit denominator.	$\frac{3}{80} = \frac{0.3}{8}$ $8\overline{)0.3\,0^{6}0^{4}0}$ with quotient $0.0\,3\,7\,5$ $\therefore \frac{3}{80} = 0.0375$

(c) $\frac{1}{6}$

If it is not obvious how to express the fraction as a decimal, use short division. When the same remainder (or patterns of remainders) appear, the resulting digits must recur.	$6\overline{)1.\,{}^{4}0\,{}^{4}0\,{}^{4}0\ldots}$ with quotient $0.\,1\,6\,6\ldots$ $\therefore \frac{1}{6} = 0.1\dot{6}$

You may notice from experience that a simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5.

If the denominator of a fraction contains no prime factors other than 2 or 5, then this number must divide into 10 or 100 or 1000 etc. meaning that the fraction can be expressed with a power of 10 denominator, and thus be expressed as a terminating decimal.

A simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5; otherwise, it can only be expressed as a recurring decimal.

3 Expressing rational numbers as decimals

For each of the following rational numbers, determine whether it must be recurring or terminating, justifying your answer.

(a) $\frac{7}{16}$

THINKING	WORKING
1 Ensure the fraction is in its simplest form, and consider the prime factors of the denominator.	The only prime factor of 16 is 2, since $16 = 2^4$. $\frac{7}{16}$ can be expressed as a terminating decimal, because the denominator contains no prime factors other than 2 or 5.
2 Evaluate the reasonableness of your answer by converting the fraction to a decimal.	$\begin{array}{r} 0.4\;\;3\;\;7\;\;5 \\ 16\overline{)\,7.0^{6}0^{12}0^{8}0} \end{array}$

(b) $\frac{2}{15}$

THINKING	WORKING
1 Ensure the fraction is in its simplest form, and consider the prime factors of the denominator.	The prime factors of 15 are 3 and 5, since $15 = 5 \times 3$. $\frac{2}{15}$ can only be expressed as a recurring decimal, because the denominator contains a prime factor other than 2 or 5, namely 3.
2 Evaluate the reasonableness of your answer by converting the fraction to a decimal.	$\begin{array}{r} 0.1\;\;3\;\;3\;\;3\ldots \\ 15\overline{)\,2.0^{5}0^{5}0^{5}0\ldots} \end{array}$

While terminating decimals can readily be converted to simplified fractions, converting a recurring decimal to a simplified fraction is not quite so obvious. The following examples illustrate one technique that can be used to accomplish this.

4 Expressing recurring decimals as fractions

Write each of the following numbers in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

(a) $0.\dot{2}$

THINKING	WORKING
1 Assign the number to a variable.	Let $x = 0.2222...$
2 Multiply by a suitable power of 10 to obtain a number with the same repeated digits following the decimal point.	$10x = 2.2222...$
3 Subtracting these two expressions should result in a rational number.	$10x - x = 2.2222... - 0.2222...$ $9x = 2$ $x = \frac{2}{9}$

(b) $0.1\overline{23}$

1 Assign the number to a variable, then multiply by a suitable power of 10 so that the recurring digits occur immediately after the decimal point.	Let $x = 0.123\,2323...$ $10x = 1.232\,323....$
2 Multiply by a suitable power of 10 to obtain a number with the same repeated digits following the decimal point.	$1000x = 123.2323...$
3 Subtracting these two expressions should result in a rational number.	$1000x - 10x = 123.\cancel{2323...} - 1.\cancel{2323...}$ $990x = 122$ $x = \frac{122}{990}$ $= \frac{61}{495}$

EXERCISE 3.1 Rational and irrational numbers

Worked Example 1

1 For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) 17 **(b)** -3 **(c)** 0 **(d)** $\frac{3}{5}$

(e) $\frac{24}{6}$ **(f)** $-4\frac{1}{3}$ **(g)** $6.\dot{3}$ **(h)** $-0.1\overline{23}$

(i) $\sqrt{16}$ **(j)** $2\sqrt{6}$ **(k)** $\sqrt{\frac{1}{4}}$ **(l)** $\sqrt[3]{6}$

(m) $\sqrt[3]{0.001}$ **(n)** π **(o)** $\sqrt{-9}$ **(p)** $\sqrt[3]{0}$

2 Express each of the following rational numbers as a terminating or recurring decimal. 2

(a) $\frac{4}{10}$ (b) $\frac{213}{100}$ (c) $\frac{3}{40}$ (d) $\frac{7}{12}$

(e) $\frac{2}{11}$ (f) $\frac{7}{11}$ (g) $\frac{11}{30}$ (h) $\frac{34}{9}$

3 For each of the following rational numbers, without calculating the decimal representation, determine whether it can be written as a terminating decimal, or only as a recurring decimal, justifying your answer. 3

(a) $\frac{5}{8}$ (b) $\frac{11}{12}$ (c) $\frac{3}{160}$ (d) $\frac{6}{70}$

4 Express each of the following numbers in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1. 4

(a) 1.8 (b) 0.14 (c) 4.111 (d) $0.\dot{3}$

(e) $0.\dot{4}$ (f) $0.\dot{1}\dot{4}$ (g) $0.\overline{125}$ (h) $0.2\dot{3}$

(i) $0.5\overline{4}$ (j) $0.3\dot{1}$ (k) $0.34\dot{7}$ (l) $0.1\overline{28}$

5 Which of the following numbers is equal to $0.1\overline{21}$?

A $0.\overline{12}$ B $0.\overline{21}$ C $0.1\dot{2}$ D $0.\overline{121}$

6 (a) Which of the following numbers is equal to $0.4\dot{9}$?

A 0.499999 B $0.\overline{49}$ C 0.5 D $0.\overline{499}$

(b) Explain the misconceptions or errors that lead to selecting the incorrect alternatives.

7 Demonstrate that $0.\dot{a}$, where $a \in \mathbb{Z}$, is equal to $\frac{a}{9}$.

8 Demonstrate that $0.\overline{ab}$, where $a, b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.

9 Demonstrate that $0.a\dot{b}$, where $a, b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$.

10 Let $E = \sqrt{11+\sqrt{x}} - \sqrt{11-\sqrt{x}}$. By considering E^2, deduce which positive integer values of x make E an integer.

3.2 The language and logic of proof

Mathematical knowledge is unlike the knowledge of other scientific disciplines. Instead of consisting of theories based on observation and evidence that can possibly be falsified, it consists of *theorems*: significant mathematical statements that have been proven to be absolutely true (think of Pythagoras' theorem, for instance). Scientific theories are similar to mathematical *conjectures*. A conjecture is a statement that mathematicians have reason to believe may be true, but which has not been proven definitively. One of the most famous mathematical conjectures is the *Goldbach conjecture*, named after the eighteenth-century mathematician Christian Goldbach:
'Every even integer greater than 2 can be expressed as the sum of two prime numbers.'

The Goldbach conjecture (see right) seems likely to be true; in fact, it has been shown to be true for every integer up to 4×10^{18}. However, it is, at present, unproven—there may, in fact, be a very large even integer that cannot be expressed as the sum of two prime numbers.

$$\begin{aligned} 4 &= 2+2 \\ 6 &= 3+3 \\ 8 &= 3+5 \\ 10 &= 3+7 = 5+5 \\ 12 &= 5+7 \\ 14 &= 3+11 = 7+7 \\ &\vdots \\ 4 \times 10^{18} &= \cdots \end{aligned}$$

The idea of mathematical proof is extremely powerful. It enables mathematics to be a rich and robust system of knowledge that will never be falsified; further, any result that has been proven can safely be used to help establish further useful results, adding to this system of knowledge.

In order to prove mathematical statements, it is important to use clear, unambiguous language and valid logic. The focus of this section is the language and logic used to construct, combine and evaluate the truth or falsehood of mathematical statements about numbers.

Note that the word 'statement' will be used to not only refer to an assertion that is true or false (such as 'The number 7 is prime' or 'All multiples of 10 are also multiples of 5'), but also to refer to an assertion involving one or more variables that becomes true or false whenever values are substituted for the variable (such as 'n is a multiple of 5' or '$x^2 < 20$').

Negating statements

Statement	Example
The negation of a mathematical statement is the statement that is true precisely when the original statement is false, and vice versa.	The negation of the statement $x > 0$ is $x \leq 0$. Notice that, for any number that is substituted in place of x, if the statement $x > 0$ is true, then the negation $x \leq 0$ will be false, and vice-versa.
The negation of a statement can be obtained by preceding the statement with the phrase: 'It is not the case that…'.	If n represents an integer, then the negation of the statement 'n is an even number' is 'It is not the case that n is an even number'. This is equivalent to saying that n is an odd number.
A negation of a statement can be written using a symbol representation. In this case the symbol '$\neg$' represents the word '*not*'.	If P represents any statement, then the negation of P can be written as $\neg P$ or simply '*not P*'.

The negation of statements involving the words 'and' or 'or' can sometimes cause confusion.	Consider negating the statement 'Either $x = 5$ or $x = 7$'. If it is not the case that x is equal to 5 or 7 then it must be the case that $x \neq 5$ *and* $x \neq 7$. As another example, consider the negation of the statement: $x > 0$ and $x < 10$. If it is not the case that x is between 0 and 10, then either $x \leq 0$ or $x \geq 10$.

The negation of 'P and Q' is '*not* P or *not* Q'. The negation of 'P or Q' is '*not* P and *not* Q'.

The preceding rules about negating 'and' and 'or' statements are known as De Morgan's laws.

5 Negating statements using De Morgan's laws

Negate the following statements.

(a) n is divisible by 2 or n is divisible by 3

THINKING	WORKING
Think about preceding the statement with the phrase, 'it is not the case that…', and then use De Morgan's laws.	n is not divisible by 2 and n is not divisible by 3. Equivalently, n is divisible by neither 2 nor 3.

(b) $x > 0$ and $x \leq 5$

Think about preceding the statement with the phrase, 'It is not the case that…', and then use De Morgan's laws.	$x \leq 0$ or $x > 5$

Statements involving quantifiers

Statement	Example
Often, mathematical statements involving variables are true for certain values of the variable and false for others.	The statement $(x-2)(x-5) = 0$ is true if x is equal to either 2 or 5, but false otherwise. But consider the following statement, where x represents a real number: $x^2 \geq 0$. This is true for every possible real number that can be substituted in place of x. Therefore, you could say: 'For all real numbers x, $x^2 \geq 0$'.
Statements that assert that some property is true for all possible values of a variable are so common that a special symbol is often used to enable such statements to be written in a more compact way. The symbol $\forall$ is known as the universal quantifier and is used as shorthand for the phrase 'for all'.	The statement could be written as follows: '$\forall$ real numbers x, $x^2 \geq 0$'. This can be condensed further by making use of the notation $x \in \mathbb{R}$ to signify that the variable x is an element of the set of real numbers. So, you could write the statement as follows: '$\forall x \in \mathbb{R}, x^2 \geq 0$'.

Sometimes you want to consider at least one possible value of the variable that makes it true and not all the values, as was mentioned before. For such statements, the symbol ∃ (known as the existential quantifier), can be used as shorthand for the phrase 'there exists'.	Consider the statement: '∃ an integer, n such that $n^2 + 2n$ is a prime number'. This can also be condensed further as follows: '$\exists n \in \mathbb{Z}$ such that $n^2 + 2n$ is a prime number'.

The symbol ∀ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol ∃ (known as the *existential quantifier*) is used to mean 'there exists'.

6 Translating statements involving quantifier symbols

Translate the following statements into everyday language. Also determine whether or not each statement is true, justifying your answer.

(a) $\exists x \in \mathbb{R}$ such that $x^2 = \sqrt{x}$.

THINKING	WORKING
1 Determine the meaning of the statement.	There is at least one real number whose square is equal to its square root.
2 In order to prove a 'there exists' statement is true, you need to be able to provide one example.	$(1)^2 = \sqrt{1}$
3 Interpret the answer.	Given that $(1)^2 = \sqrt{1}$, the statement is true. There exists at least one real number for which $x^2 = \sqrt{x}$.

(b) ∀ integers n, the number $5n$ is even.

1 Determine the meaning of the statement.	The product of any integer n and the number 5 results in an even number.
2 In order to prove a 'for all' statement is false, you need to provide a single counter-example. That is, an example demonstrating falsehood.	Let $n = 3$. $3 \times 5 = 15$
3 Interpret the answer.	For $n = 3$, $5n = 15$ which is not an even number, so the statement is false. There may be integers that, when multiplied by 5, result in an even number, however, this does not apply in every case ($\forall n$).

Providing a single example is always sufficient to prove that a 'there exists' statement is true.

Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

Examples and counter examples

In part (a) of Worked Example 6, it was claimed that the statement '$\exists x \in \mathbb{R}$ such that $x^2 = \sqrt{x}$' is true, and to justify this claim, a single example was provided of a real number whose square was equal to its square root (namely, $x = 1$). Providing a single example is always sufficient to prove that a 'there exists' statement is true.

In part (b) of Worked Example 6, it was claimed that that the statement '$\forall$ integers n, the number $5n$ is even' is false, and to justify this claim, a single example was provided of an integer (namely, $n = 3$) which, when multiplied by 5, gives a number that is not even (namely, 15). An example such as this, that demonstrates the falsehood of a statement, is known as a counter-example. Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

7 Writing statements with quantifier symbols

Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, determine whether or not each statement is true, justifying your answer.

(a) The square-root of any positive integer is less than or equal to the integer.

THINKING	WORKING
1 Identify the quantifier.	'Of any' specifies that for all positive integers. So the quantifier is $\forall$.
2 Create a mathematical statement.	For all positive integers n, the square root of n is less than or equal to n. $\forall n \in \mathbb{Z}^+, \sqrt{n} \leq n$
3 Determine whether the $\forall$ 'for all' statement can be proved to be false using a counter-example.	For $n = 1$ $\sqrt{1} \leq 1$ For $n = 2$ $\sqrt{2} \leq 2$ This could continue for $n = 1, 2, 3, 4$ and 5. There is not an obvious choice for a counter-example and the true statements cannot be calculated for infinite positive integers. It is assumed to be true, but has not been proved to be so.

(b) There is at least one real number that, when squared, results in a smaller number.

1 Identify the quantifier.	'There is at least one' specifies the existence of a value. So the quantifier is $\exists$.
2 Create a mathematical statement.	$\exists n \in \mathbb{R}$ such that $n^2 < n$.

3 Determine whether the ∃ 'there exists' statement can be proved to be correct by calculating an example.	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ $\therefore \left(\frac{1}{2}\right)^2 < \frac{1}{2}$ This statement is true as shown by the existence of a single example: $n = \frac{1}{2}$.

Note that the symbols ∀ and ∃ may be used together in a single statement, however, the order in which they appear is important. As an example, consider the following two statements:

- ∀ integers n, ∃ an integer m such that $n + m$ is a multiple of 5.
- ∃ an integer n, such that ∀ integers m, $n + m$ is a multiple of 5.

The first statement is true as it says that for every integer, you can find another integer to add to it to give a sum that is a multiple of 5.

The second statement is false as it says there is a special integer that has the property that when you add any other integer to it, you always obtain a multiple of 5.

When the symbols ∀ and ∃ appear together in the same statement, the order in which they appear is important.

Negating statements involving quantifiers

Consider the negation of the statement '∀ real numbers x, $x^2 \geq 0$'. If it is not the case that the square of every real number is greater than or equal to zero, then it must mean that there is at least one real number whose square is less than zero. Thus, the negation is '∃ a real number x such that $x^2 < 0$'. Notice how the negation of a 'for all' statement is a 'there exists' statement. The reverse is also true. For example, the negation of the statement: '∃ a real number x such that $3x = x$' is '∀ real numbers x, $3x \neq x$'.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

8 Negating statements involving quantifiers

Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer.

(a) ∀ integers n, $2n$ is even

THINKING	WORKING
1 Interpret the meaning of the statement.	For all integers, n, multiplying any integer by 2 will result in an even number.
2 Write a negating statement. The negation of a 'for all' statement is a 'there exists' statement.	There exists an integer, n such that $2 \times n$ is not even (odd). $\exists n \in \mathbb{Z}$ such that $2n$ is not even.
3 Determine whether the original or the negation is true. Justify your answer.	The original statement is true. By definition, an even number is any integer value that is a multiple of 2.

(b) $\exists$ a real number x, such that $x^2 = -1$.

1 Interpret the meaning of the statement.	There exists a real number x, where $x^2 = -1$.
2 Write a negating statement. The negation of a 'there exists' statement is a 'for all' statement.	For all real numbers x, $x^2 \neq -1$. $\forall x \in \mathbb{R}, x^2 \neq -1$
3 Determine whether the original or the negation is true. Justify your answer.	The negation is true. The square root of any real number is greater than or equal to 0 and therefore is not equal to -1.

(c) $\exists$ an integer n, such that n is even and n is prime.

1 Interpret the meaning of the statement.	There exists a real number n, where n is both even and prime.
2 Write a negating statement. The negation of a 'there exists' statement is a 'for all' statement.	For all real numbers n, n is neither even nor prime. $\forall n \in \mathbb{Z}$, n is odd or not prime.
3 Determine whether the original or the negation is true. Justify your answer.	The original statement is true. This is proven by the single example where $n = 2$. Two is the only even and prime number. Every other even number is a multiple of 2 and therefore not prime.

Conditional statements

Consider the following statement: if n is a multiple of 10, then n is an even number. This is an example of a conditional statement. A conditional statement (also known as an 'if–then' statement, or an 'implication') is one that asserts that *if* some condition holds, *then* it must be the case that some property is true. Conditional statements are so common in mathematics that there is a variety of ways to express them. The given statement could be represented in any of the following ways:

- If n is a multiple of 10, then n is an even number.
- n is an even number if n is a multiple of 10.
- n being a multiple of 10 is a *sufficient* condition to conclude that n is even.
- n being even is *necessary* if n is a multiple of 10.
- n is a multiple of 10 *implies that* n is an even number.

Finally, the implication symbol, $\Rightarrow$, is used to mean 'implies that'. Thus, the statement could be also written as follows: n is a multiple of 10 $\Rightarrow$ n is an even number.

Each of the following means the same as $P \Rightarrow Q$:

- If P, then Q.
- Q if P.
- P is a *sufficient* condition to conclude that Q.
- Q is *necessary* if P.
- P *implies that* Q.

9 Using the implication symbol

Rewrite the following conditional statements using the implication symbol, $\Rightarrow$.

(a) If n ends in a zero, where $n \in \mathbb{Z}$, then n is even.

THINKING	WORKING
If P, then Q can be written as $P \Rightarrow Q$.	If n ends in a zero $\Rightarrow$ n is even.

(b) $n > 3$ is a sufficient condition to conclude that n is positive.

'P is a sufficient condition to conclude Q' means the same as 'If P, then Q'.	If $n > 3$, then n is positive. $n > 3 \Rightarrow n$ is positive.

(c) $n > 3$ is necessary if n is greater than 4.

'P is a necessary condition if Q' means the same as 'If Q, then P'.	If $n > 4$ then $n > 3$. $n > 4 \Rightarrow n > 3$

The converse, contrapositive and negation of a conditional statement

The converse

The *converse* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol. For example, consider the conditional statement previously introduced:

- original: n is a multiple of 10 $\Rightarrow$ n is an even number
- converse: n is an even number $\Rightarrow$ n is a multiple of 10

Notice that the converse is not saying the same thing as the original. The original statement is claiming that if a number is a multiple of 10, then it must be even (which is true). But the converse is claiming that if a number is even, then it must be a multiple of 10 (which is definitely not true.)

For a statement of the form $P \Rightarrow Q$ that involves some variable, the converse is the statement $Q \Rightarrow P$.

The contrapositive

The *contrapositive* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol, and also negating both statements. Again, consider the conditional statement previously introduced:

- original: n is a multiple of 10 $\Rightarrow$ n is an even number
- contrapositive: n is not an even number $\Rightarrow$ n is not multiple of 10

As a classic illustrative real-life example, the contrapositive of the statement, 'If an animal is a poodle, then it is a dog' is 'If an animal is not a dog, then it is not a poodle'. Notice, again, how the original and the contrapositive statements are essentially saying the same thing.

For $P \Rightarrow Q$, the contrapositive is in the form *not* $P \Rightarrow$ *not* Q.

The negation of a conditional statement

Now consider the *negation* of the statement 'n is a multiple of 10 $\Rightarrow$ n is an even number'. This statement is essentially saying that for every integer that is a multiple of 10, this integer must also be even. If this were not the case, it would mean that there must exist some integer that is a multiple of 10 but is not even. Using the real-life example from earlier, the negation of 'If an animal is a poodle, then it is a dog' would be 'There exists some animal which is a poodle, but not a dog'.

The negation of a conditional statement of the form $P \Rightarrow Q$ that involves some variable is 'There exists some value of the variable for which P is true, but Q is false'.

WARNING

Be aware that the negation of a conditional statement is different from both the converse and the contrapositive.

10 Writing the converse, contrapositive and negatiown of a conditional statement

Consider the following conditional statement: 'If n is a perfect square, then n is divisible by 3'.

(a) Write the statement in the form $P \Rightarrow Q$ and determine whether the statement is true or false.

THINKING	WORKING
1 Write the conditional statement in the form $P \Rightarrow Q$.	$\forall n$, n is a perfect square $\Rightarrow$ n is divisible by 3.
2 A 'for all' statement can be proven false by providing a counter-example.	The perfect squares: $\{1, 4, 9, 16, 25\ldots\}$ There exists a value of n for which n is a perfect square but is not divisible by 3. A counter-example: $n = 16$ is a perfect square but is not divisible by 3.
3 Interpret the answer.	The conditional statement, 'If n is a perfect square, then n is divisible by 3', is false.

(b) Write the converse to the conditional statement and determine whether the converse statement is true or false.

1 For $P \Rightarrow Q$, the converse is in the form $Q \Rightarrow P$.	$\forall n$, n is divisible by 3 $\Rightarrow$ n is a perfect square.
2 A 'for all' statement can be proven false by providing a counter-example.	The multiples of 3: $\{3, 6, 9, 12, 15\ldots\}$ There exists a value of n for which n is divisible by 3 but is not a perfect square. A counter-example: 12 is divisible by 3 but is not a perfect square.
3 Interpret the answer.	The converse statement, 'If n is divisible by 3, then n is a perfect square', is false.

(c) Write the contrapositive to the conditional statement and determine whether the contrapositive statement is true or false.

1 For $P \Rightarrow Q$, the contrapositive is in the form *not* $P \Rightarrow$ *not* Q.	$\forall n$, n is not a perfect square $\Rightarrow$ n is not divisible by 3.

2 A 'for all' statement can be proven false by providing a counter-example. Alternatively, recall that the contrapositive of a conditional statement is true whenever the original statement is true.	Not a perfect square: $\{2, 3, 5, 6, 7\ldots\}$ There exists a value of n for which n is not a perfect square but is divisible by 3. A counter example: 6 is a perfect square but is divisible by 3.
3 Interpret the answer.	The contrapositive statement, 'If n is not a perfect square, then n is not divisible by 3', is false.

(d) Write the negation to the conditional statement and determine whether the negation statement is true or false.

1 For $P \Rightarrow Q$, the negation is in the form 'There exists some value of the variable for which P is true, but Q is false'.	$\exists n$, n is a perfect square but n is not divisible by 3.
2 A 'there exists' statement can be proven true by providing one example.	Perfect squares: $\{1, 4, 9, 16, 25\ldots\}$ There exists a value of n for which n is a perfect square, but not divisible by 3. An example: 4 is a perfect square but is not divisible by 3.
3 Interpret the answer.	The negation statement, 'There exists some value of n, where n is a perfect square not divisible by 3', is true.

Logical equivalent statements

Recall that the conditional statement 'n is a multiple of $10 \Rightarrow n$ is an even number' is true, however, its converse 'n is an even number $\Rightarrow n$ is a multiple of 10' is not. Sometimes, however, a conditional statement and its converse are both true. As an example, notice that if $x = 5$ then $2x = 10$ and, conversely, if $2x = 10$, then $x = 5$. This means that for the two statements, $x = 5$ and $2x = 10$, whenever one of these is true, the other must also be true. Such statements are said to be *logically equivalent*.

Two statements are logically equivalent if whenever one is true, the other must also be true.

There is a variety of ways to represent the fact that $x = 5$ and $2x = 10$ are logically equivalent:

- $x = 5$ is *necessary and sufficient* for $2x = 10$
- $x = 5$ *if and only if* $2x = 10$
- $x = 5 \Rightarrow 2x = 10$ and $2x = 10 \Rightarrow x = 5$

Finally, the symbol $\Leftrightarrow$ is often used to denote logical equivalence. Thus, $x = 5 \Leftrightarrow 2x = 10$ (or equivalently, $2x = 10 \Leftrightarrow x = 5$).

Each of the following can be used to express the fact that P and Q are logically equivalent:

- P is *necessary and sufficient* for Q.
- P *if and only if* Q.
- $P \Rightarrow Q$ and $Q \Rightarrow P$
- $P \Leftrightarrow Q$

11 Using the logical equivalence symbol

Rewrite the following statement using the logical equivalence symbol, $\Leftrightarrow$: 'For n to be divisible by 5, it is both necessary and sufficient that n end in either 0 or 5'.

THINKING	WORKING
'P is a necessary and sufficient condition for Q' means that P and Q are logically equivalent.	n is divisible by 5 $\Leftrightarrow$ n ends in 0 or 5.

EXERCISE 3.2 The language and logic of proof

Worked Example

1 Determine the negation of each of the following statements. 5

(a) P and Q are both even.
(b) $x > 5$ or $x < -5$
(c) x is divisible by either 7 or 8.
(d) $x = 0$ or $y = 0$

2 Translate the following statements into everyday language. Also determine whether or not the statement is true, justifying your answer where appropriate. 6

(a) $\forall$ integers n, the number $2n + 3$ is odd.
(b) $\exists$ a real number x such that $\frac{1}{x} = x$.
(c) $\forall$ real numbers x, $x^2 > 0$.
(d) $\exists x \in \mathbb{R}$ such that $x^2 = -1$.
(e) $\forall n \in \mathbb{Z}$, the number $n(n + 1)$ is divisible by 3.
(f) $\forall$ real numbers x and y, $x - y > 0$.
(g) $\forall$ real numbers x, $\exists$ a real number y such that $x + y = 0$.
(h) $\exists$ a real number x, such that $\forall$ real numbers y, $xy = y$.

3 Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, determine whether or not the statement is true, justifying your answer where appropriate. 7

(a) The square of any integer is greater than the integer.
(b) There is a real number that, when multiplied by 5 gives an answer of 0.
(c) The sum of any two consecutive integers is odd.
(d) There is a real number equal to its square.
(e) The sum of the squares of any two real numbers is less than the product of the numbers.
(f) There is a special real number with the property that whenever another real number is divided by it, this other real number is obtained.
(g) Every integer is divisible by at least one integer.

4 Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer where appropriate. 8

(a) $\forall$ real numbers x, $x^2 > 0$.
(b) $\exists$ a real number x such that $x^2 = x$.
(c) $\forall$ positive integers n, $10n > n$.
(d) $\forall$ real numbers x, x is either positive or negative.
(e) $\exists$ an integer n such that $n \neq 0$ and $n^2 < 1$.
(f) $\forall$ integers n, either $(-1)^n = 1$ or $(-1)^n = -1$.

9 **5** Rewrite the following statements using the implication symbol, $\Rightarrow$.

(a) If $x > 3$, then $x^2 > 9$.

(b) If n is divisible by 9, then n is divisible by 3.

(c) $n > 5$ implies that $n > 4$.

(d) $7P$ is positive if $P > 3$.

(e) Q is even is necessary if $2Q$ is a perfect square.

(f) m is a multiple of 6 is a sufficient condition to conclude that m is divisible by 3.

(g) It is necessary that $x^2 > 2$ if $x < -2$.

(h) n is even and greater than 2 is a sufficient condition to conclude that n is not prime.

10 **6** Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate.

(a) If n is divisible by 20, then n is divisible by 5.

(b) If n is divisible by 3, then n^2 is divisible by 3.

(c) If $x > 7$, then $10x > 70$.

(d) If $xy = 0$, then either $x = 0$ or $y = 0$.

(e) If n is divisible by 5, then the final digit of n is 5.

(f) If $x = 4$ and $y = 4$, then $xy = 16$.

(g) If n is divisible by 24, then n is even and n is divisible by 3.

11 **7** Rewrite the following statements using the logical equivalence symbol, $\Leftrightarrow$.

(a) n is even if and only if n^2 is even.

(b) $x + y = 0$ if and only if $x = -y$.

(c) n being even and divisible by 3 is necessary and sufficient for n to be divisible by 6.

8 For each of the following statements, provide three examples that are consistent with the statement, and then a single counter-example to prove that the statement is actually false.

(a) If a positive integer is divisible by 7, then it is not divisible by 3.

(b) $\forall x \in \mathbb{R}, x^2 \geq x$

(c) If P is a prime number, then $2^P - 1$ is prime.

(d) If a positive integer is divisible by both 10 and 6, then it is also divisible by 60.

(e) Suppose that x and y are real numbers. If $x > 3$, then $x^2 - 2y > 5$.

(f) Suppose that x and y are positive real numbers. Then $xy > x + y$.

9 (a) Which of the following statements is true?

A $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 6$.

B $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, xy = 6$.

C $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x + y = 6$.

D $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 6$.

(b) Why are the alternatives incorrect?

10 What is the negation of the statement $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 6$?

A $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y \neq 6$.

B $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x + y \neq 6$.

C $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \neq 6$.

D $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y \neq 6$.

11 Write the negation of the following statement, where x represents a real number:
$x > 0$ and $x < 10 \Rightarrow x \geq 0$ and $x \leq 10$. Also, determine whether the original statement or the negation is true.

12 Consider the following conjecture: Start with any positive integer. If the integer is even, halve it. If the integer is odd, triple it and add one. Repeat this process. Eventually, the integer 1 will be obtained. This is known as the '$3x + 1$' conjecture. It is yet to be proved, but it has been shown to be true for all integers up to roughly 10^{14}. Verify this conjecture for the following positive integers:

(a) 6

(b) 13

(c) 7

3.3 Methods of proof

Aside from examples and counter-examples (which can be used to prove the truth of a 'there exists' statement or the falsehood of a 'for all' statement), a mathematical proof typically consists of a sequence of statements with each statement following directly from either definitions, previous steps or other established results. In this section, several common strategies for constructing proofs are illustrated.

As many of the statements proved are concerned with even and odd numbers, and divisibility more generally, it is necessary to be familiar with the following definitions:

An integer n is said to be even if n can be expressed in the form $n = 2k$, for $k \in \mathbb{Z}$.

For $k = 5$, $n = 2 \times 5 = 10$

Therefore, the number 10 is said to be even, because it is the product of 2 and an integer, 5.

Similarly, an integer n is said to be odd if n can be expressed in the form $n = 2k + 1$, for $k \in \mathbb{Z}$.

For $k = 5$, $n = 2 \times 5 + 1 = 10 + 1 = 11$

Therefore, the number 11 is said to odd, because it is one more than the product of 2 and an integer, 5.

Finally, an integer is said to be divisible by an integer m, if n can be expressed in the form $n = mk$ for $k \in \mathbb{Z}$.

For $m = 3$, $n = 3 \times k = 3k$

Therefore, consider the set of numbers $k = \{1, 2, 3, 4\ldots\}$ then $n = \{3, 6, 9, 12\ldots\}$, where n represents the set of integers divisible by 3.

Direct proof

The most straightforward way to prove a statement is to use a direct proof. A direct proof typically starts by introducing any relevant variables, and assuming any given condition holds, and then establishes the desired result via a logical sequence of valid statements. Note that if the statement to be proved has the form 'If P, then Q', then you assume that P is true, and then proceed to prove that Q must be true.

12 Direct proof

Use a direct proof to prove that if a number is odd, then its square is also odd.

THINKING	WORKING
1 Define a numerical variable, and assume the information given—in this case, that the number is odd.	Let p be an odd integer.
2 Use the fact that an odd number can always be expressed in the form $2k + 1$.	Then $p = 2k + 1$ for some integer k.
3 Aim to demonstrate that the statement is true—in this case, that p^2 is odd. Thus, obtain an expression for p^2 and demonstrate that it can be expressed in the form 2(integer) + 1.	$p^2 = (2k+1)^2$ $= 4k^2 + 4k + 1$ $= 2(2k^2 + 2k) + 1$ As $2k^2 + 2k$ must be an integer, it follows that p^2 is odd.

Proof by contraposition

Recall that the contrapositive of the statement $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. As the contrapositive is logically equivalent to the original statement, $P \Rightarrow Q$, the original statement can be proved indirectly by proving $\neg Q \Rightarrow \neg P$; that is, by assuming that Q is false, and then proceeding to prove that P must be false.

13 Proof by contraposition

Use a contrapositive proof to prove that if $5n + 3$ is odd, then n is even.

THINKING	WORKING
1 Determine the contrapositive statement.	The contrapositive statement is: if n is not even, then $5n + 3$ is not odd. This means, if n is odd, then $5n + 3$ is even.
2 Introduce appropriate variables and proceed to prove the contrapositive statement.	Let n be an odd integer. Then $n = 2k + 1$ for some integer k. $5n + 3 = 5(2k + 1) + 3$ $= 10k + 5 + 3$ $= 10k + 8$ $= 2(5k + 4)$ As $5k + 4$ must be an integer, it follows that $5n + 3$ is even if n is odd.
3 Write your conclusion.	Because n is odd, then $5n + 3$ is even. Therefore, by the contrapositive statement, if $5n + 3$ is odd, then n is even.

Proof by contradiction

Another form of indirect proof, but one that is not restricted to proving conditional statements, is 'proof by contradiction'. The basic idea of such a proof is to assume that the statement needing to be proved is *false*, and then demonstrate that this assumption leads to an absurd and impossible result. This then must mean that the initial assumption—that the statement was false—cannot be true, meaning that the original statement must be true!

Two proofs by contradiction are illustrated in the following example. The first is a famous proof that $\sqrt{2}$ is irrational. The second is a proof of the statement from the previous worked example, only using contradiction instead of contraposition.

14 Proof by contradiction

Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{2}$ is irrational.

THINKING	WORKING
1 Assume the negation of the statement to be proved.	Suppose, for a contradiction, that $\sqrt{2}$ is rational.
2 Recall that a rational number can be expressed in the form $\frac{p}{q}$ for integers p and q, with $q \neq 0$, and with p and q having no common factor other than 1.	Then $\sqrt{2} = \frac{p}{q}$ for integers p and q, with $q \neq 0$, and with p and q having no common factor other than 1.
3 Using a series of valid steps, arrive at a contradiction.	$2 = \frac{p^2}{q^2}$ $p^2 = 2q^2$, hence p^2 is divisible by 2. If p^2 is divisible by 2, then p is divisible by 2. Therefore $p = 2m$ for some integer m. $4m^2 = 2q^2$ $2m^2 = q^2$ Hence q^2 is also divisible by 2, so q is divisible by 2. Since p and q are both divisible by 2 then there is a contradiction since p and q have no common factors other than 1.
4 Interpret your result.	It follows that $\sqrt{2}$ is an irrational number.

(b) If $5n + 3$ is odd, then n is even.

1 Assume the negation of the statement to be proved.	Suppose, for a contradiction, that $5n + 3$ is odd, but n is also *odd*.
2 Using a series of valid steps, arrive at a contradiction.	Then $n = 2k + 1$ for some integer k. $5n + 3 = 5(2k + 1) + 3$ $= 10k + 5 + 3$ $= 10k + 8$ $= 2(5k + 4)$ As $5k + 4$ must be an integer, it follows that $5n + 3$ is even if n is odd. But this is a contradiction, as the number $5n + 3$ is odd, and a number cannot be both even and odd.
3 Interpret your result.	Hence, the number n cannot be odd, and so must be even.

Note that the proof in part **(a)** of the previous example used the fact that if the square of an integer is divisible by 2, then the original integer must also be divisible by 2. This fact is true not just for the number 2 but for any integers with no perfect square factors other than 1.

For example, if k^2 is even then k is also even:

$$\begin{aligned} k^2 &= 36 \\ k &= \pm\sqrt{36} \\ &= \pm 6 \end{aligned}$$

where k is even because it can be expressed in the form $k = 2m$, $-6 = 2 \times -3$ or $6 = 2 \times 3$.

This stems from the fact that, for an even number, $k = 2m$, where $m \in \mathbb{Z}$ then:

$$\begin{aligned} k^2 &= (2m)^2 \\ &= 4m^2 \\ &= 2(2m^2) \end{aligned}$$

You may use this fact when modifying the previous proof to prove the irrationality of other surds in the subsequent exercise.

You may have noticed that the contradiction proof in part **(b)** looks very similar to the contraposition proof from Worked Example 13. The actual logic used is almost identical—the difference is that there is no actual contradiction obtained in the contrapositive proof since it was never assumed that $5n + 3$ was odd. Note also that a contradiction proof does not aim for a *particular* contradiction—any contradiction is sufficient. In a situation such as this, when a contradiction proof is very similar to a contrapositive proof, the contrapositive proof is considered more efficient and elegant.

Proving logical equivalences

The simplest method to prove a statement of the form $P \Leftrightarrow Q$ is to separately prove both $P \Rightarrow Q$ and $Q \Rightarrow P$, as demonstrated in the following example.

15 Proving logical equivalences

Let n be a positive integer. Prove that $n + 9$ is odd if and only if $n - 8$ is even.

THINKING	WORKING
1 First, suppose that $n + 9$ is odd. Then $n + 9 = 2k + 1$ for some integer k.	$n - 8 = (n+9) - 17$ $= (2k+1) - 17$ $= 2k - 16$ $= 2(k-8)$ Thus, $n - 8$ is even if $n + 9$ is odd.
2 Conversely, suppose that $n - 8$ is even. Then $n - 8 = 2k$ for some integer k.	$n + 9 = (n-8) + 17$ $= 2k + 17$ $= 2k + 16 + 1$ $= 2(k+8) + 1$ Thus, $n + 9$ is odd if $n - 8$ is even.

The final example presented is a more complex proof of a well-known divisibility result.

16 Proving divisibility results

Prove that a three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

THINKING	WORKING
1 Define the variables for the digits and write an expression for the number.	Let a, b, c be the digits, in order, of a three-digit number, N. The actual number is therefore $N = 100a + 10b + c$.
2 Since the statement involves 'if and only if', it is necessary to split it into two conditional proofs. First suppose that the number is divisible by 3 and demonstrate that this implies that the sum of the digits is divisible by 3.	First suppose that N is divisible by 3. Thus, $100a + 10b + c = 3k$ for some integer k. Then $99a + 9b + a + b + c = 3k$ $a + b + c = 3k - 99a - 9b$ $a + b + c = 3(k - 33a - 3b)$ Thus, the sum of the digits is divisible by 3.
3 Finally, demonstrate the converse of the previous step to complete the proof.	Conversely, suppose the sum of the digits is divisible by 3. Then $a + b + c = 3k$ for some integer k. Then $N = 100a + 10b + c$ $= 99a + 9b + a + b + c$ $= 99a + 9b + 3k$ $= 3(33a + 3b + k)$ Thus, N is divisible by 3.

EXERCISE 3.3 Methods of proof

Worked Example

12 **1** Use a direct proof to prove each of the following statements.

(a) The sum of any two odd integers is even.

(b) The sum of an odd integer and an even integer is always odd.

(c) The product of two odd integers is odd.

(d) The sum of two consecutive odd numbers is divisible by 4.

(e) The sum of the squares of five consecutive integers is divisible by 5.

(f) The product of two rational numbers is rational.

(g) The sum of two rational numbers is rational.

(h) If n is odd, then n^2 is odd.

(i) If n is divisible by 7, then n^2 is divisible by 7.

(j) If $m + n$ and $n + p$ are even, where m, n, p are integers, then $m + p$ is even.

13 **2** Use a contrapositive proof to prove each of the following statements.

(a) Let n be an integer. If $3n + 2$ is even, then n is even.

(b) If a and b are integers and ab is even, then at least one of a and b is even.

(c) Let n be an integer. If $n^3 + 5$ is odd, then n is even.

(d) If x is irrational then $\sqrt{x}$ is irrational.

(e) If x is irrational, then $\frac{1}{x}$ is irrational.

14 **3** Use a proof by contradiction to prove each of the following statements.

(a) $\sqrt{3}$ is irrational.

(b) $\sqrt{5}$ is irrational.

(c) The sum of a rational and an irrational number is irrational.

(d) The product of a rational and an irrational number is irrational.

(e) There are no integers a and b such that $18a + 6b = 1$.

15 **4** Prove each of the following logical equivalences.

(a) Let n be a positive integer. $n + 9$ is even if and only if $n + 6$ is odd.

(b) Let n be a positive integer. $n - 3$ is odd if and only if $n + 2$ is even.

(c) Let n be a positive integer. n is even if and only if $13n + 4$ is even.

(d) Let n be a positive integer. n is odd if and only if $7n + 6$ is odd.

(e) Let n be a positive integer. n is even if and only if n^2 is even.

5 Consider the following statement: 'If two integers have an even product, then at least one of the two integers must be even'.

(a) To prove this statement by contraposition, it would be necessary to:

A suppose that at least one of the two integers is even, and then show that the product must be even.

B suppose that at least one of the integers is odd, and then show that the product must be odd.

C suppose that both integers are odd, and then show that the product must be odd.

D suppose that the two integers have an even product and that both integers are odd, and then show that a contradiction arises.

(b) Explain the misconceptions or errors that lead to selecting the incorrect alternatives.

6 Prove that the number $1 - 5\sqrt{2}$ is irrational.

7 Let a, b, c be positive real numbers such that $ab = c$. Prove that $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.

8 Prove that if $a^2 - 2a + 7$ is even, then a is odd.

Worked Example 16

9 Prove that a four-digit number is divisible by 9 if and only if the sum of its digits is divisible by 9.

10 (a) Prove that for all real numbers a and b, $a^2 + b^2 \geq 2ab$.

(b) Using the result from part (a) or otherwise, prove that for all positive real numbers a and b, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

(c) Using the result from part (a) or otherwise, prove that for all positive real numbers a and b, $\frac{a+b}{2} \geq \sqrt{ab}$.

11 Prove that every odd integer can be expressed as the difference between two perfect squares.

12 Prove that if a and $b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$.

13 Let k be a positive integer. Prove that if $2^{k+2} + 3^{3k}$ is divisible by 5, then $2^{k+3} + 3^{3k+3}$ is also divisible by 5.

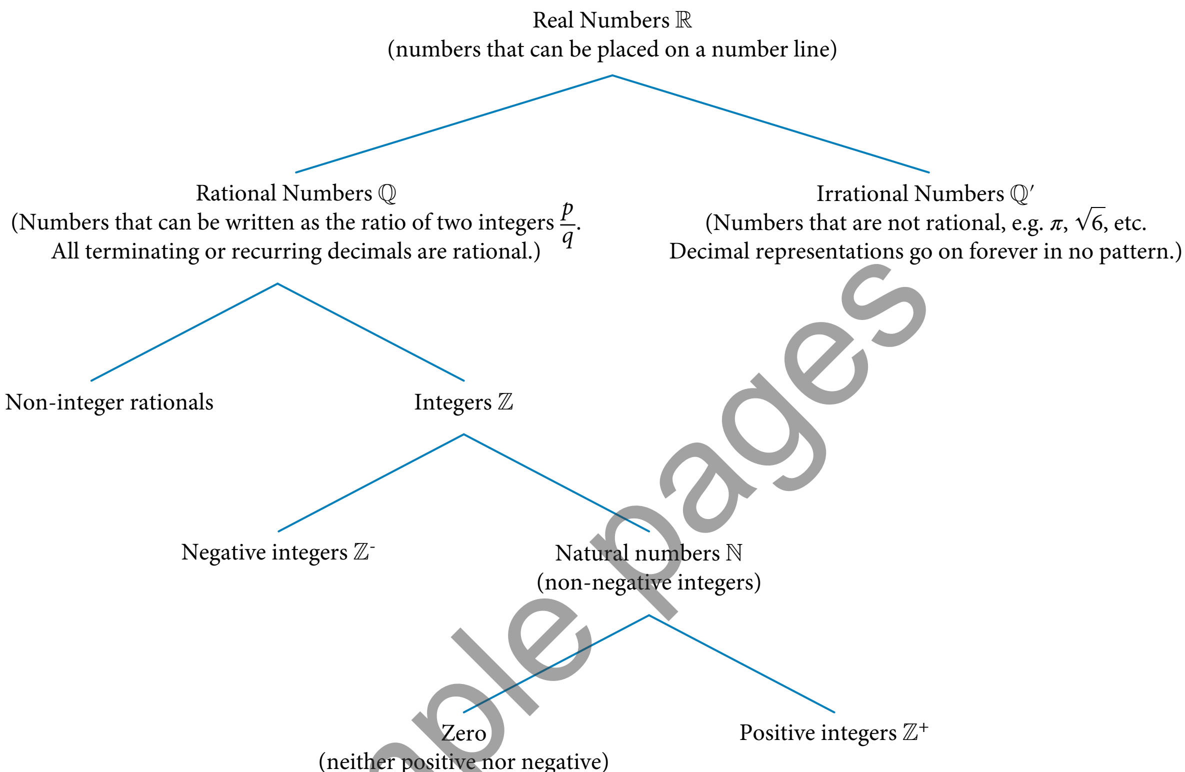
14 It is known that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square. Use this result to help prove that $\sqrt{6} + \sqrt{10}$ is irrational.

15 Use a proof by contradiction to demonstrate that there is no rational solution to the equation $x^3 + x + 1 = 0$.
Hint: Start by supposing, for a contradiction, that $r = \frac{p}{q}$ is a rational solution to the equation, where p and q are integers with no common factor other than 1 and with $q \neq 0$. Then consider what would happen if both p and q were odd, or if one of them was even and the other odd.

3 Summary

Real numbers

The diagram below summarises the different types of real numbers.



A simplified fraction can be expressed as a terminating decimal if the denominator has no prime factors other than 2 or 5; otherwise, it can only be expressed as a recurring decimal.

When converting a recurring decimal to a simplified fraction, multiply by the power of 10 equal to the number of digits that are recurring.

Negating 'and' and 'or' statements

The negation of 'P and Q' is '*not* P or *not* Q'.

The negation of 'P or Q' is '*not* P and *not* Q'.

Quantifiers

The symbol ∀ (known as the *universal quantifier*) is used to mean 'for all'.

The symbol ∃ (known as the *existential quantifier*) is used to mean 'there exists'.

Providing a single example is always sufficient to prove that a 'there exists' statement is true.

Providing a single counter-example is always sufficient to prove that a 'for all' statement is false.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

Conditional statement

Each of the following means the same as $P \Rightarrow Q$:

- If P, then Q.
- Q if P.
- P is a *sufficient* condition to conclude that Q.
- Q is *necessary* if P.
- P *implies that* Q.

For a statement of the form $P \Rightarrow Q$ that involves some variable:

- The converse is the statement $Q \Rightarrow P$.
- The contrapositive is the statement *not* $Q \Rightarrow$ *not* P.
- The negation is the statement 'there exists some value of the variable for which P is true, but Q is false'.

Logically equivalent statements

Two statements are logically equivalent if whenever one is true, the other must also be true.

Each of the following can be used to express the fact that P and Q are logically equivalent:

- P is *necessary and sufficient* for Q.
- P *if and only if* Q.
- $P \Rightarrow Q$ and $Q \Rightarrow P$.
- $P \Leftrightarrow Q$.

Methods of proof

A *direct proof* starts by assuming any given conditions hold, and then proceeds directly to the desired result.

A *contrapositive proof* is a direct proof of the contrapositive of a conditional statement. As this involves proving a statement that is not the same, but is logically equivalent to the original, it is known as an indirect proof.

A *proof by contradiction* involves assuming that the statement needing to be proved is *false* and then demonstrating that this assumption leads to an absurd and impossible result. This then must mean that the assumption was false, meaning that the initial statement must be true.

The simplest method to prove a statement of the form $P \Leftrightarrow Q$ is to separately prove both $P \Rightarrow Q$ and $Q \Rightarrow P$.

Chapter review 3

Exercise 3.1

1 For each of the following numbers, determine which sets from $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}'$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the number is an element of.

(a) $\frac{6}{11}$ (b) $-5.\overline{83}$ (c) -3 (d) 0 (e) $\sqrt{10}$

Exercise 3.1

2 Express each of the following rational numbers as a terminating or recurring decimal.

(a) $\frac{17}{100}$ (b) $\frac{11}{40}$ (c) $\frac{5}{12}$

Exercise 3.1

3 Write each of the following numbers in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common factor other than 1.

(a) 0.38 (b) $0.\dot{5}$ (c) $0.2\overline{31}$

Exercise 3.1

4 Which one of the following numbers can be written as a terminating decimal?

A $\frac{47}{150}$ B $\frac{2}{900}$ C $\frac{47}{320}$ D $\frac{55}{140}$

Exercise 3.2

5 Determine the negation of each of the following statements.

(a) n is divisible by either 5 or 6. (b) $0 \leq x \leq 10$ (c) p is prime and $p \neq 2$.

Exercise 3.2

6 Translate the following statements into everyday language. Also determine whether or not the statement is true, justifying your answer where appropriate.

(a) $\forall$ integers n, the number $10n$ is divisible by 4.

(b) $\exists$ a real number x such that $x^2 = 2$.

(c) $\forall$ real numbers x, $\exists$ a real number y such that $xy = 1$.

Exercise 3.2

7 Determine the negation of each of the following statements. Also determine whether the original statement or the negation is true, justifying your answer where appropriate.

(a) $\exists$ a real number x such that $x^2 < 0$.

(b) $\forall$ integers n, $5n + 3$ is odd.

(c) $\forall$ integers n, $\exists$ an integer m such that $2m = n$.

Exercise 3.2

8 Rewrite the following statements using the implication symbol, $\Rightarrow$.

(a) $3x$ is negative if $x < -10$.

(b) n is a multiple of 5 is necessary if n ends in a zero.

(c) n is a multiple of 4 is a sufficient condition to conclude that n is even.

(d) $x > 3$ is a sufficient condition to conclude that $x^2 > 9$.

9 Write the converse, the contrapositive and the negation of each of the following conditional statements. Determine whether each of the original statements, converse, contrapositive and negation is true or false, justifying your answer where appropriate. Exercise 3.2

(a) If n is divisible by 4, then n is divisible by 12.

(b) If $xy = 1$, then either $x = 1$ or $y = 1$.

(c) If $x > 0$ and $x < 3$, then $x^2 < 9$.

10 Which of the following statements is true? Exercise 3.2

A $x = 3 \Leftrightarrow x^2 = 9$ **B** $x < 3 \Leftrightarrow x < 4$ **C** $x = 3 \Leftrightarrow x^3 = 27$ **D** $x > 1 \Leftrightarrow 2x > 1$

11 Use a direct proof to prove that the sum of the squares of any two odd integers is even. Exercise 3.3

12 Use a contrapositive proof to prove that if $n^3 - n$ is not divisible by 4, then n must be even. Exercise 3.3

13 Use a proof by contradiction to prove that $\sqrt{7}$ is irrational. Exercise 3.3

14 Use a proof by contradiction to prove that for all integers a and b, $12a - 9b \neq 20$. Exercise 3.3

15 Prove that n is odd if and only if n^3 is odd. Exercise 3.3

16 Demonstrate that $0.a\overline{3b}$, where a and $b \in \mathbb{Z}$, can be expressed in the form $\frac{p}{q}$ where p and $q \in \mathbb{Z}$, $q \neq 0$. Exercise 3.1

17 Prove that if a, b, c and d are consecutive integers with $a < b < c < d$, then $-a^2 + b^2 - c^2 + d = a + b + c + d$. Exercise 3.3

18 Prove that $\sqrt[3]{2}$ is irrational. Exercise 3.3

19 Prove that if a and $b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$. Exercise 3.3

20 Prove that if $6^k + 4$ is divisible by 5, then $6^{k+1} + 4$ is divisible by 10. Exercise 3.3

21 Prove that for all positive real numbers x and y, $\frac{1}{7} + \frac{x}{y} \neq \frac{1+x}{7+y}$. Exercise 3.3

22 Prove that a four-digit number is divisible by 7 if and only if the number obtained by removing the final digit and subtracting it twice from the original number is divisible by 7. Exercise 3.3

23 Prove that if a, b, c are odd integers, then the equation $ax^2 + bx + c$ cannot possibly have a rational solution. Exercise 3.3

24 Solve the following: $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$. Hint: Consider modifying the method used to convert a recurring decimal to a fraction. Exercise 3.1

25 Prove that if $P = \dfrac{\left(\frac{1}{3}\right)^n + 1}{\left(\frac{1}{3}\right)^n + 3}$, then $\dfrac{1}{4-3P} = \dfrac{\left(\frac{1}{3}\right)^{n+1} + 1}{\left(\frac{1}{3}\right)^{n+1} + 3}$. Exercise 3.3