Pythagoras' theorem

2

Did Pythagoras 'steal' the theorem that bears his name? One of the world's most famous mathematical statements is not named after its discoverer.

Several ancient civilisations, including the Babylonians, Chinese and Egyptians, knew about the special relationship between the sides of a right-angled triangle. Pythagoras of Samos (believed to have lived 580–496 BCE) was a Greek philosopher who probably learnt about this theorem from Egyptian engineers known as the 'rope stretchers'. They used a circle of rope with 12 knots tied at regular intervals around the circle. Pegging the rope to the ground at intervals of 3, 4 and 5 knots produced a right-angled triangle, which was used to ensure that building foundations or walls could be constructed accurately.

So, why does Pythagoras get all the credit? Possibly because he led a secretive cult devoted to the study of mathematics, with their motto being 'all is number'.

Although Pythagoras did not discover the theorem, it is possible that he was the first to produce a proof of the theorem using geometry. The photograph on the opposite page is of a statue of Pythagoras on the island of Samos, in Greece.

Forum

Why do you think Pythagoras and his followers held secret meetings and kept their discoveries to themselves?

Think of some real-life objects that can be made of or be described by right-angled triangles.

Why learn this?

Pythagoras' theorem is a useful mathematical tool applied by builders, carpenters, engineers and surveyors to determine the straight-line distance between two points or to check whether an angle is a right angle. It is also used in the audio-visual industry to determine the height and width of TV and projection screens.

After completing this chapter you will be able to:

- use Pythagoras' theorem to determine whether a triangle is right-angled
- use Pythagoras' theorem to find side lengths in right-angled triangles
- apply Pythagoras' theorem to practical situations
- complete simple calculations using surds
- express irrational answers in exact (surd) form
- identify Pythagorean triples.

Recall 2

Equipment required: calculator for Questions **2–4, 6**

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, you can download a Recall Worksheet from the eBook or the Pearson Places website.

1 Round each of the following numbers to 2 decimal places.

(a) 45.789 **(b)** 12.2311 **(c)** 4.549 567 835 6

2 Use your calculator to find the following, correct to 2 decimal places where appropriate.

(a) 12^2 **(b)** 55^2 **(c)** 37.5^2

(d) $3^2 + 4^2$ **(e)** $9^2 + 13^2$ **(f)** $2.5^2 + 7.1^2$

3 Use your calculator to find the exact values of each of the following.

(a) $\sqrt{81}$ **(b)** $\sqrt{169}$ **(c)** $\sqrt{256}$

4 Use your calculator to find the following. Round your answers to 2 decimal places.

(a) $\sqrt{65}$ **(b)** $\sqrt{658}$ **(c)** $\sqrt{321.45}$

5 Solve the following equations.

(a) $x + 5 = 11$ **(b)** $144 + x = 225$ **(c)** $45 - x = 12$

6 Solve the following for positive values of x. Where necessary, express your answer as a decimal correct to 2 decimal places.

(a) $x^2 = 36$ **(b)** $x^2 = 39$ **(c)** $x^2 = 325$

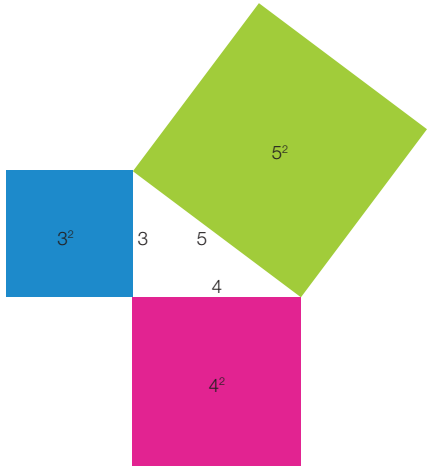
(d) $x^2 + 6 = 70$ **(e)** $x^2 - 20 = 101$ **(f)** $\frac{x^2}{4} = 325$

Exploration Task

You can download this activity from the eBook or the Pearson Places website.

Pythagorean triples

In this activity, you will explore the unusual pattern of Pythagoras' theorem solutions where the hypotenuse is a whole number.



Pythagorean triples

2.5

A group of three positive whole numbers (a, b, c) that satisfy Pythagoras' theorem is called a **Pythagorean triple** or **Pythagorean triad**. Some common Pythagorean triples are shown in the table below.

(3, 4, 5)	(8, 15, 17)	(12, 35, 37)
(5, 12, 13)	(9, 40, 41)	(16, 63, 65)
(7, 24, 25)	(11, 60, 61)	(20, 21, 29)

Multiples of a Pythagorean triple must also satisfy Pythagoras' theorem. For example, if you multiply each number in (3, 4, 5) by 2, you have (6, 8, 10) and $6^2 + 8^2 = 10^2$. Dividing Pythagorean triples by a number also gives a set of numbers that satisfy Pythagoras' theorem. For example, dividing (7, 24, 25) by 2 gives (3.5, 12, 12.5) and $3.5^2 + 12^2 = 12.5^2$.

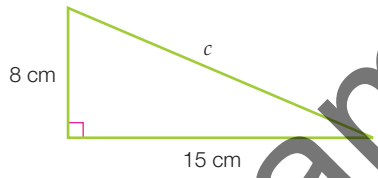
The numbers in a Pythagorean triple must satisfy Pythagoras' theorem exactly. Numbers that only work when they are approximate or rounded are not exact enough to be considered Pythagorean triples.

Worked example 10

W.E. 10

Using your knowledge of common Pythagorean triples, state the value of the unknown side in each of the following triangles.

(a)



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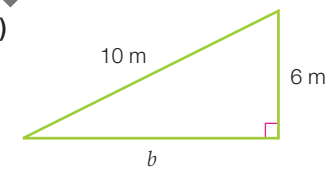


Thinking			Working	
(a)	**1**	Check if the values of the shorter sides match a common triple, (a, b, c).	(a)	$a = 8, b = 15, c = ?$
	2	State the answer.		$c = 17$ cm
(b)	**1**	Check if the values of the hypotenuse and one of the shorter sides match a common triple, (a, b, c).	(b)	$a = 6$ and $c = 10$ $= 3 \times 2$ $= 5 \times 2$ (6, ?, 10) is a multiple (× 2) of the common triple (3, 4, 5).
	2	Find the missing number in the triple by multiplying the common triple number by the same multiple.		$b = 4 \times 2$
	3	State the answer.		$b = 8$ m

2.5 Pythagorean triples

Navigator

Answers p. 634

1, 2 (columns 1, 2), 3 (a–c), 5, 6 (a–d), 7, 8, 13, 14	1, 2 (columns 2, 3), 5, 6 (a–d), 7, 8, 9, 10, 11, 13, 14	1, 2 (column 3), 4, 5, 6 (e–h), 7, 8, 9, 10, 11, 12, 13, 14

Fluency

1 **(a)** (3, 4, 5) is a Pythagorean triple. Fill in the missing numbers in these sentences about Pythagorean triples which are multiples of (3, 4, 5).

To make (9, 12, 15) you multiply each number by _____.
To make (21, 28, 35) you multiply each number by _____.
To make (60, 80, 100) you multiply each number by _____.

(b) (7, 24, 25) is a Pythagorean triple. Write the new Pythagorean triples made when the following are applied.

The numbers are multiplied by 10.
The numbers are divided by 2.
The numbers are divided by 10.

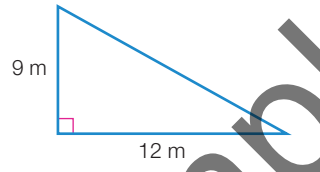
(c) (5, 12, 13) is a Pythagorean triple. Fill in the missing value in each Pythagorean triple made from (5, 12, 13).

(10, 24, ___), (___, 6, 6.5), (30, ___, 78)

W.E. 10

2 Using your knowledge of Pythagorean triples, find the value of the missing sides in each of the following.

(a)



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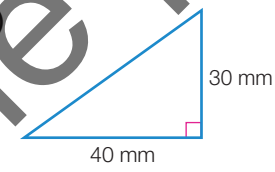

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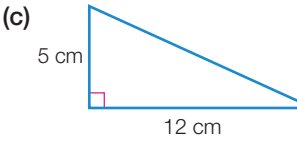

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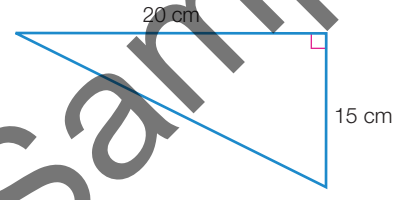

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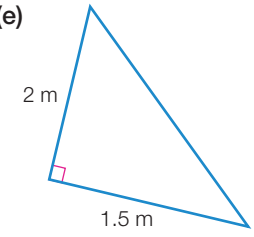

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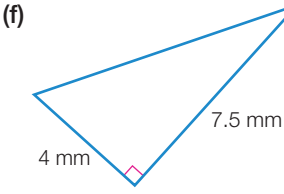

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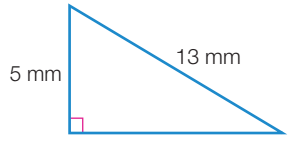
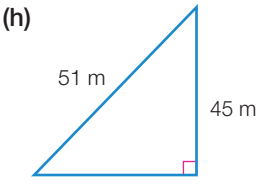
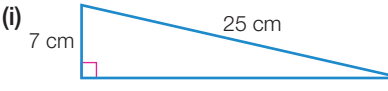


(h)



(i)



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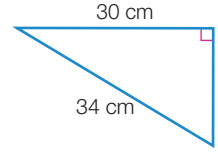

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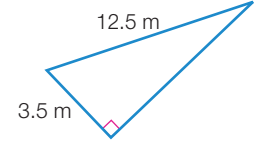
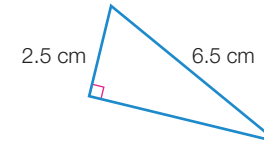


(l)



3 Find the value of c, using known Pythagorean triples.

(a) $a = 10, b = 24$ **(b)** $a = 15, b = 36$ **(c)** $a = 9, b = 12$

(d) $a = 27, b = 36$ **(e)** $a = 18, b = 80$ **(f)** $a = 50, b = 120$

(g) $a = 25, b = 60$ **(h)** $a = 500, b = 1200$ **(i)** $a = 80, b = 84$

4 If $a = 9$, $b = 40$ and $c = 41$ is a Pythagorean triple, which of the following is also a Pythagorean triple?

A (36, 160, 165) **B** (18, 80, 82) **C** (55, 240, 246) **D** (90, 405, 410)

5 If (5.6, 9, 10.6) satisfies Pythagoras' theorem, another set of numbers that also satisfies Pythagoras' theorem is:

A (65, 90, 106) **B** (10.2, 18, 1.06) **C** (14, 22.5, 26.5) **D** (14, 2.25, 20)

Understanding

6 Determine which of the following are Pythagorean triples.

(a) (6, 8, 10) **(b)** (24, 45, 51) **(c)** (14, 48, 50) **(d)** (25, 60, 80)

(e) (10, 16, 28) **(f)** (20, 48, 52) **(g)** (10, 18, 22) **(h)** (12, 50, 53)

7 Toula has just baked a cake in the shape of a rectangle. She wants to divide the cake into two right-angled triangles by cutting along a diagonal of the cake. Toula is then going to place a cake ribbon around the outside of each of the two triangular pieces of cake. The length of the cake is 32 cm and its width is 24 cm. Use a Pythagorean triple to determine exactly how much cake ribbon she will need.

8 **(a)** Complete the table, using only odd values for m.

(b) What do you notice about the numbers in each row of the table?

(c) Beginning with 'Take any odd number', describe the process used to generate the Pythagorean triples in the table.

m	$\frac{m^2-1}{2}$	$\frac{m^2+1}{2}$
a	b	c
3	4	5
5	12	13
7		
9		
11		

9 **(a)** Complete the table, using consecutive odd numbers for x and y.

(b) What do you notice about the numbers in the last three columns?

(c) Beginning with 'Take any two consecutive odd numbers', describe the process used to generate the Pythagorean triples in the table.

x	y	$x + y$	xy	$\sqrt{a^2 + b^2}$
		a	b	c
1	3	4	3	5
3	5	8	15	17
5	7			
7	9			
9	11			

Reasoning

10 (48, 55, 73) is a Pythagorean triple.

If the numbers in the Pythagorean triple (7, 24, 25) are doubled, the new triple is (14, 48, 50).

If the numbers in the Pythagorean triple (16, 63, 65) are multiplied by 3, the new triple is (48, 189, 195).

Here are some Pythagorean triples with at least one number that is 24 or less. If a whole-number triple including the number 48 can be made, write the new triple. Write 'no' if it is not possible.

(a) (3, 4, 5) **(b)** (5, 12, 13) **(c)** (8, 15, 17)

(d) (9, 40, 41) **(e)** (11, 60, 61) **(f)** (12, 35, 37)

(g) (13, 84, 85) **(h)** (24, 143, 145)

11 Explain how the Pythagorean triple (160, 384, 416) can be obtained from another Pythagorean triple.

12 Derek believes he has found a rule for Pythagorean triples. He says 'In each Pythagorean triple with no common factor, it appears that there is one even number and two odd numbers, one number with a factor of 5 and at least one prime number'.

3, 4, 5	8, 15, 17	20, 21, 29
5, 12, 13	12, 35, 37	28, 45, 53
7, 24, 25	16, 63, 65	36, 77, 85
9, 40, 41	20, 99, 101	44, 117, 125
11, 60, 61	24, 143, 145	52, 165, 173
13, 84, 85	28, 195, 197	

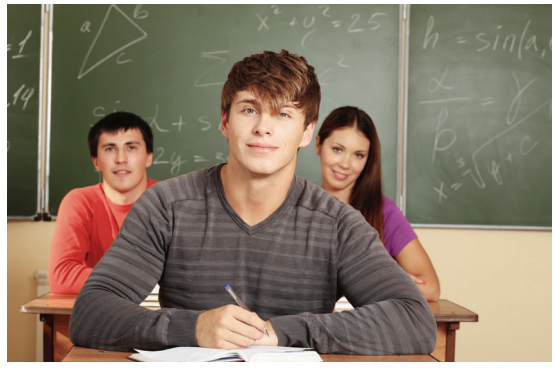
From the given sets of Pythagorean triples:

(a) Find at least four triples that satisfy Derek's rule.

(b) Find at least one triple that does not satisfy Derek's rule.

(c) If you could modify Derek's rule to make it more accurate, how would you change it?

(d) Can you find the pattern in each column and then write the next triple in each column?



Open-ended

13 Choose one of the Pythagorean triples from the list on page 118 and list its multiples (a, b, c) that have a, b and c less than 300.

14 **(a)** Stefan told Gabriella that for a particular Pythagorean triple (a, b, c), the length a *is* a multiple of 4 and b is a multiple of 5. What is one Pythagorean triple that satisfies this information?

(b) Write two multiples of the answers to **(a)**.

Problem solving

How old was Pythagoras?

Use the clues about the Pythagorean triple (a, b, c), to find an estimate of how many years Pythagoras lived:

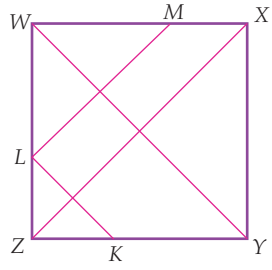
- c is Pythagoras' age, a value less than 100
- c is bigger than b and their difference is a cube number
- c is bigger than a and their difference is a square number
- a is a square number
- b is bigger than a and their difference is 41
- the digits of b are the same and are both odd numbers.

Strategy options

- Guess and check.
- Work backwards.
- Test all possible combinations.

Challenge 2

1 $WXYZ$ is a square of side 3 units. K is a point on YZ, L is a point on ZW and M is a point on WX. $KL \,||\, YW$ and $LM \,||\, ZX$. Let $ZK = a$ and $WL = b$ units.



(a) Write an equation for b in terms of a and rearrange to find $a + b$.

(b) What is the exact length of KL in terms of a?

(c) What is the exact length of LM in terms of b?

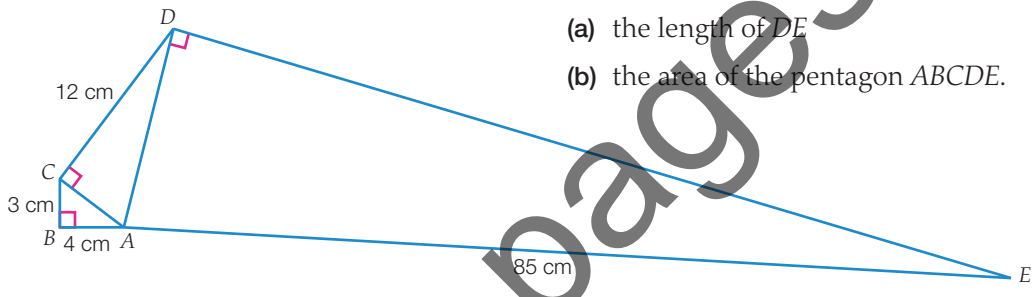
(d) What is the exact length of $KL + LM$?

2 Triangles ABC, ACD and ADE have a right angle at B, C and D, respectively. $AB = 4$ cm, $BC = 3$ cm and $CD = 12$ cm.

If $AE = 85$ cm, find:

(a) the length of DE

(b) the area of the pentagon $ABCDE$.



3 Write a whole number whose square root is between 7 and 8.

4 What is the smallest number that, when multiplied by 2016, gives a square number?

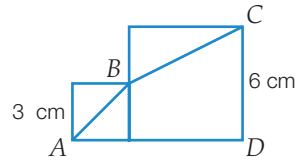
5 If m, n are integers and $m > n \geq 1$, then the numbers $m^2 - n^2$, $2mn$, $m^2 + n^2$ are Pythagorean triples (that is, a set of three numbers that obey Pythagoras' theorem).

(a) Show that $(m^2 + n^2)^2 = (m^2 - n^2)^2 + (2mn)^2$.

(b) Find the Pythagorean triples, with no common factors, whose smallest member is:

(i) 3 **(ii)** 5 **(iii)** 7 **(iv)** 8 **(v)** 9

6 A square of side length 3 cm is placed next to a square of side length 6 cm as shown. The perimeter of the quadrilateral $ABCD$ is:



A $3\sqrt{2}$ cm **B** $3\sqrt{5}$ cm

C $6\sqrt{2}$ cm **D** $3(5 + \sqrt{2} + \sqrt{5})$ cm

Chapter review 2

Maths literacy

converse	proof	rational approximation	theorem
hypotenuse	Pythagoras' theorem	right-angled triangle	
irrational number	Pythagorean triple	surd	

Copy and complete the following using the words and phrases from this list, where appropriate. A word or phrase may be used more than once.

1 The longest side of a right-angled triangle is called the _____.

2 For any _____, the square of the length of the _____ is equal to the sum of the squares of the lengths of the two shorter sides.

3 The equation $c^2 = a^2 + b^2$ is known as _____.

4 A _____ is a logical, step-by-step argument that demonstrates the truth of a mathematical statement.

5 A mathematical statement that has been proven true is known as a _____.

6 When a number such as $\sqrt{84}$ is written as a rounded decimal, it's called a _____.

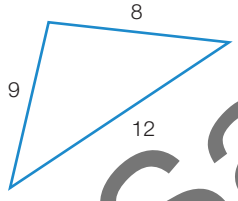
7 An irrational number written in exact form, such as $\sqrt{5}$, is known as a _____.

8 A group of three whole numbers that satisfy Pythagoras' theorem, such as (3, 4, 5), is called a _____.

Fluency

1 Use the converse of Pythagoras' theorem to determine whether the following triangles are right-angled. **2.1**

(a)



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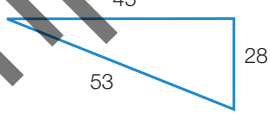

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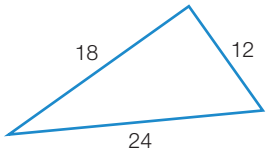
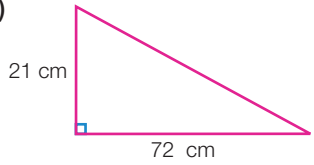


2 Find the length of the hypotenuse in the following right-angled triangles. Write your answers in exact form. **2.2**

(a)



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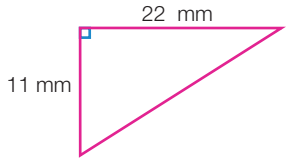

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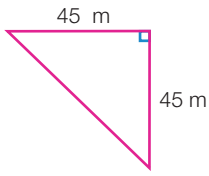
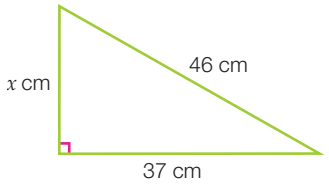


3 Use Pythagoras' theorem to find the value of x. Write your answer correct to 2 decimal places if necessary. **2.3**

(a)



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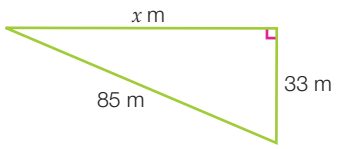

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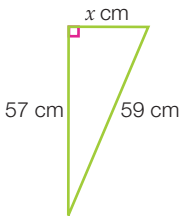
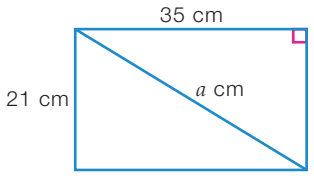
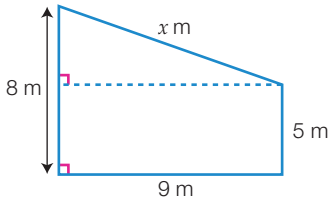


4 Use Pythagoras' theorem to find the value of the unknown in each of the following diagrams. Write your answer correct to 2 decimal places if necessary. **2.2, 2.3**

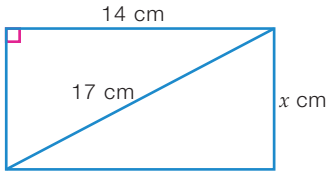
(a)



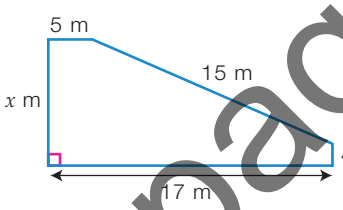
(b)



(c)



(d)



5 The Leaning Tower of Pisa was originally 150 m tall, but is leaning over so that a stone dropped from the top lands 5 m from the base. How far vertically does the stone fall? **2.4**

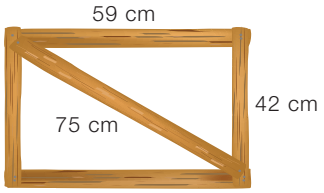
A 149.92 m **B** 150.00 m **C** 159.98 m **D** 162.00 m

6 Find the value of c in each of the following Pythagorean triples. **2.5**

(a) $a = 12, b = 16, c = ?$ **(b)** $a = 45, b = 200, c = ?$ **(c)** $a = 132, b = 385, c = ?$

Understanding

7 Simon is making the base for a dog's kennel. He put in a diagonal brace for extra support. **2.1**

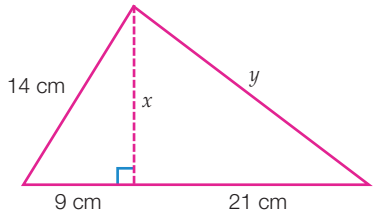


(a) From the measurements shown, has Simon built the base 'square'?

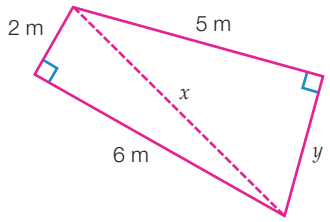
(b) What would be a reasonable measurement for the diagonal for Simon to consider the base to be 'square'?

8 Find the exact value of the variables in the following diagrams. **2.2, 2.3**

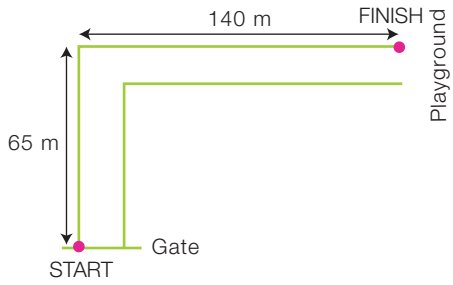
(a)



(b)

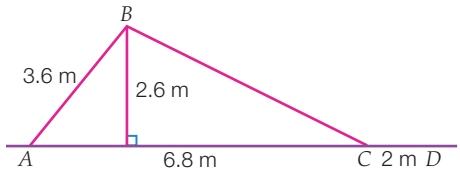


9 Ming and Tong decided to have a race from the gate of the park to the playground. Ming has decided to run along the left edge of the path and can do so at a speed of 7 metres per second. However, Tong decides to go directly from the gate to the playground, but this is very sandy and he can only run at 5 metres per second. Who wins the race to the playground?



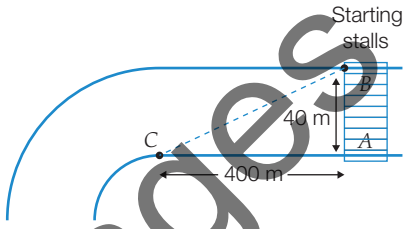
2.2, 2.4

10 A playground slide is 2.6 m high. The ladder is 3.6 m long. The distance from the bottom of the slide, *C*, to the base of the ladder, *A*, is 6.8 m. The end of the slide, *CD*, is 2 m. Find the total length of the slide from *B* to *D*.



2.2, 2.3

11 Before each horse race, a barrier draw determines each horse's starting position. Closest to the rails is the best position and is represented by point *A* on the diagram shown. The least preferred position is near the outside edge of the track (point *B*). It can be assumed that all jockeys aim to take their horses in a direct line to the beginning of the first corner (point *C*).



2.2, 2.4

(a) How much further does the horse at point *B* travel than the horse at point *A*?

(b) Is this difference significant? How might this affect the race?

Reasoning

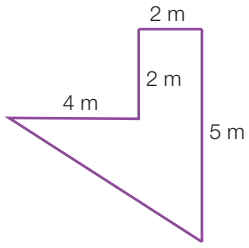
12 State whether or not each of the following is a Pythagorean triple.

2.5

(a) (120, 350, 370) **(b)** (16, 30, 35) **(c)** (77, 420, 427)

13 Daniel and Courtney are going to make a garden in the backyard of their new house. They have measured the space and it is shown in the diagram.

2.4

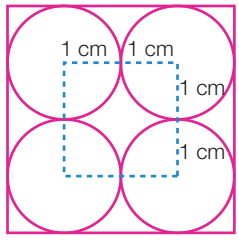


(a) Calculate the perimeter of the garden and find the number of 2 m long sleepers they will need to form the perimeter of the garden.

(b) If 2 m long sleepers cost $35 each, how much will it cost to place sleepers all around the garden bed?

14 Four steel discs, each with radius 1 cm, are placed in a square frame as shown.

2.4



(a) Find the length of the sides of this frame.

(b) In order to brace the frame, a diagonal support is to be added to the frame. Find the length of this support.

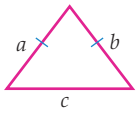
(c) Find the length of material needed to build the frame, including the diagonal brace.

Numeracy practice 2

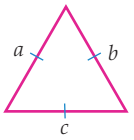
Non-calculator

1 Which triangle will satisfy Pythagoras' theorem?

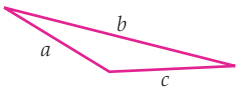
A



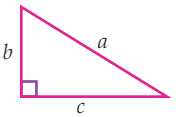
B



C



D



2 Choose the side lengths that form a right-angled triangle.

A (20, 21, 23) **B** (20, 48, 52) **C** (6, 7, 8) **D** (5, 7, 10)

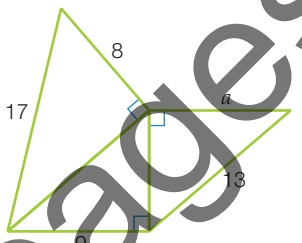
3 Which Pythagorean triple can be obtained from (36, 105, 111)?

A (12, 35, 37) **B** (36, 110, 333) **C** (24, 52.5, 55.5) **D** (72, 110, 222)

4 What is the value of the variable a?

A 5 **B** 7

C 9 **D** 11



Calculator allowed

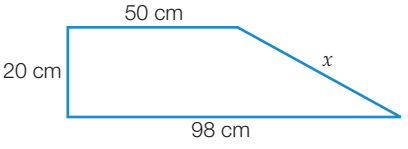
5 The two shorter sides of a right-angled triangle are of length 14 mm and 48 mm, respectively. The length of the hypotenuse is:

A 25 mm **B** 34 mm **C** 46 mm **D** 50 mm

6 The hypotenuse of a right-angled triangle is 8.70 cm and one of its shorter sides is of length 6.30 cm. The length of the other shorter side is:

A 6.00 cm **B** 6.52 cm **C** 7.14 cm **D** 8.96 cm

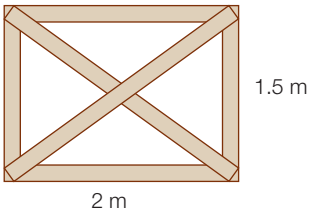
7 What is the value of the variable in the diagram shown on the right?



A 42 cm **B** 48 cm

C 52 cm **D** 64 cm

8 Jemima wants to build a rectangular timber gate of width 2 m and height 1.5 m with two diagonal supports. What is the minimum total length of timber required to build the frame of the gate?



A 6 m **B** 7 m

C 9.5 m **D** 12 m

Mixed review A

Fluency

1 After investing $8000 into a savings account paying 6.5% interest p.a., Andrew had $8390 in his account. For how many months (correct to the nearest month) was it invested? **1.5**

2 Calculate the following: **1.1**

(a) 12.5% of 450 m **(b)** 38% of 208 kg **(c)** $5\frac{1}{2}$% of $3500

3 A triangle has side lengths of 51 cm, 140 cm and 141 cm. Determine if the triangle is right-angled. **2.1**

4 Convert the following to percentages, rounding answers to 1 decimal place: **1.1**

(a) 34 goals out of 47 shots

(b) four blond-haired students out of a class of 23

(c) nine rotten apples in a crate containing 140.

5 **(a)** A TV priced at $1649 is discounted by 15%. Find the new price of the TV. **1.2**

(b) A smartphone has been discounted down to a sale price of $699. If a 10% discount was applied, what was the original price of the phone? Answer to the nearest dollar.

6 Use the following table to calculate how much income tax should be paid on the taxable incomes given below. **1.4**

Taxable income	Tax on this income
$0–$18 200	Nil
$18 201–$37 000	19c for each $1 over $18 200
$37 001–$80 000	$3572 plus 32.5c for each $1 over $37 000
$80 001–$180 000	$17 547 plus 37c for each $1 over $80 000
$180 001 and over	$54 547 plus 45c for each $1 over $180 000

(a) $5500 **(b)** $35 000 **(c)** $54 000

7 Marc works part-time at the cinema. He receives a basic hourly rate of $16.82 for any hours worked between 9 am and 9 pm. He receives time-and-a-half for any hours worked after 9 pm and any hours on Saturday. He receives double time for any hours on a Sunday. Calculate Marc's wage for the hours worked in the following week. **1.3**

Tuesday	Wednesday	Thursday	Saturday	Sunday
11 am–9 pm	2 pm–10 pm	3:30 pm–11:30 pm	12 pm–6 pm	2 pm–9 pm

8 For each of the following household items, calculate: **1.7**

(i) the weekly energy consumption, in kWh

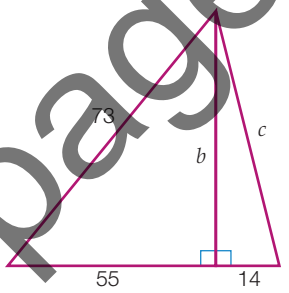
(ii) the weekly running cost, given an electricity tariff of 19.2 cents per kWh.

(a) A 100 W television, in use for 4.5 hours per day.

(b) A 75 W fridge, in use for the whole week.

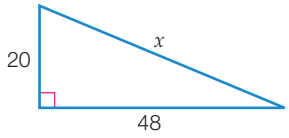
9 Kazim is a real estate agent. He is paid a $650 retainer per week, plus 2.5% of the value of any houses he sells. Calculate Kazim's income for a week in which he sold a house for $415 000. **1.3**

10 Find the values of b and c. **2.2, 2.3**



11 Find the value of the unknown side in each triangle. Give your answer in exact form. **2.2, 2.3**

(a)



(b)

Understanding

12 Mustafa has 8 shots at the basketball ring and gets 7 in. His friend Adam has 11 shots and gets 9 in. Calculate the number of goals as a percentage of the successful shots for each boy to determine who is the more accurate shooter. **1.1**

13 Richard inspects old buildings to see whether they need support to keep them standing straight up. He believes that an old house is beginning to lean, so he stands at a point on the ground 3 m from the base of a wall, which is 2.3 m high. He measures the distance from this point to the top of the wall and finds that it is 3.5 m. **2.5**

(a) Draw a diagram of a triangle to represent the situation, labelling it with all known information.

(b) Use Pythagoras' theorem to determine whether the wall is at 90° to the ground.

14 Snooker Joe hits the cue ball. The ball travels 16 cm, strikes the side, rebounds at 90° to its original direction and travels a further 63 cm before coming to rest. What is the distance between the start and finish position of the ball? **9.4**

15 After investing $5000 in a savings account for 8 months, Michael received $5347. What was the rate of interest, assuming a constant rate of simple interest? **1.5**

16 An aircraft leaves Melbourne airport and flies 37 km due east. It then changes direction and flies 684 km due north to land at Waratah airport. What is the direct distance between the two airports? **2.4**

Reasoning

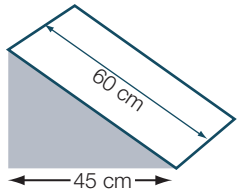
17 A second-hand store buys a large crate of DVDs for $4.70 each. It places a 250% mark-up on them and puts them out for sale. Later on, the store discounts the marked price of the DVDs by 23%. **1.2**

(a) What was the original selling price of the DVDs?

(b) What was the discounted price, rounded to the nearest 5 cents?

(c) What percentage mark-up does the new, discounted price represent, correct to the nearest whole number?

18 A ramp has the dimensions shown. It is placed against a raised platform that is 35 cm high. Does the ramp provide smooth access to the platform? **2.4**



19 The cross-section of a water jug can be approximated by the shape shown. What is its height, correct to 2 decimal places? **2.4**