NSW
STAGE 6

NEW SENIOR

Mathematics

ADVANCED YEAR 11 | 4TH EDITION

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CHAPTER 9
Probability

9.1 SETS AND SET NOTATION

A set is a collection of objects, called the elements of the set.

The set consisting of the numbers 2, 4, 6, 8 is written using a pair of {} as {2, 4, 6, 8}.

It is usually given a name using an upper-case letter.

Thus $A = \{2, 4, 6, 8\}$ is a set containing 4 elements. It is a **finite** set as the number of elements may be counted.

$n(A) = 4$ is the notation used to say that set A has 4 elements. It may also be written as $|A| = 4$.

Consider $B = \{1, 3, 5, 7, \ldots\}$. $n(B)$ is undefined as set B is an **infinite** set as the ellipses indicates that the pattern of members in the set continues without bound.

A set may have no members, $\phi = \{\ \}$, and is called the **empty** set. The Greek symbol ϕ is used to name the empty set.

Note that the set $C = \{0\}$ is not an empty set as $n(C) = 1$.

Consider the set $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and the set $A = \{2, 4, 6, 8\}$.

The set A is part of the set U, and is called a **subset** of U.

The set $\tilde{A} = \{1, 3, 5, 7\}$ and the set A, when combined make up the set U.

$\tilde{A}$ is called the **complement** of the set A as it contains all the other elements in U which were not in A.

There are several notations that may be used for the complement of a set, they are $\tilde{A}$, A' or A^C.

$\tilde{A}$ is also a subset of U.

Consider the sets $D = \{1, 2, 3, 4, 5, 6\}$ and $E = \{4, 5, 6, 7, 8\}$.

These sets have 3 elements in common, the numbers 4, 5, 6.

Let $F = \{4, 5, 6\}$ represent these numbers.

Set F is called the **intersection** of sets D and E as it contains the elements that are common to both D and E.

The notation used for the intersection of sets D and E is $\boldsymbol{D \cap E}$.

Form the set $G = \{1, 2, 3, 4, 5, 6, 7, 8\}$ which contains all the elements that are in D or E or both D and E.

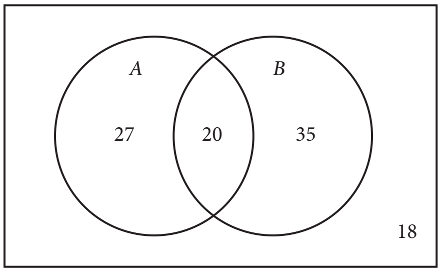
The set G is called the **union** of sets D and E and the notation used is $\boldsymbol{D \cup E}$.

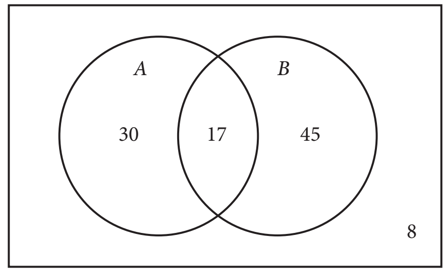
If two sets have no elements in common, they are called **disjoint** sets. Their intersection is the empty set.

Consider $L = \{3, 6, 9\}$ and $M = \{8, 10, 12\}$. $L \cap M = \{\ \} = \phi$ so, L and M are disjoint sets. Also, A and $\tilde{A}$ are disjoint sets.

Summary of terms

- In a finite set the number of elements can be counted.
- In an infinite set the number of elements cannot be counted.
- $x \in A$ says that x is an element of (or a member of) set A.
- $\phi = \{\}$ is the empty set.
- $n(A) = |A|$ represents the number of elements in the finite set A.
- U is called the universal set.
- $\tilde{A}$, A' or A^C represent the complement of set A.
- A is a subset of B if all the elements of set A come from set B, written $A \subset B$.
- $A \cap B$ represents the intersection of A and B, that is those elements common to A and B.
- $A \cup B$ represents the union of A and B, that is those elements in A or B or both A and B.
- If $A \cap B = \phi$ then A and B are disjoint sets.

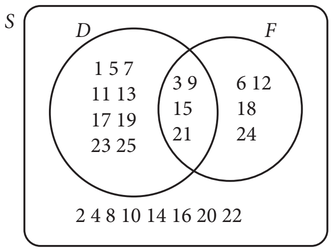




Venn Diagrams

Venn diagrams are a visual way to represent sets of data. You can name the sets with letters and talk about various properties of the sets. Venn diagrams are an efficient tool to be used when analysing data to find the probability of events.

If $S = \{1, 2, 3, 4, \ldots 25\}$ (i.e. whole numbers from 1 to 25), $D = \{1, 3, 5, \ldots 25\}$ (i.e. odd numbers from 1 to 25), $F = \{3, 6, 9, \ldots 24\}$ (i.e. multiples of 3 between 1 and 25), then $n(S) = 25$, $n(D) = 13$ and $n(F) = 8$, where the notation $n(S)$ means the number of members of set S. This information can be shown on a Venn diagram as follows:



The outer group contains all the members of set S; the inner group on the left contains all the members of set D; the inner group on the right contains all the members of set F. The area where the inner groups overlap contains the members that are in both sets D and F. The region outside the groups of D and F contain all the members of set S that are in neither set D nor set F.

To enter information onto a Venn diagram like this it is best to first find the members common to both sets D and F, then put the rest of the data into D and F. The remaining members of S that were in neither D nor F can then be put into the region outside D and F.

The set made by combining sets D and F is called the **union** of the two sets, written $D \cup F$. It consists of all the elements that are in D or in F or in both D and F:

$D \cup F = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19, 21, 23, 24, 25\}$

The set made up of the elements common to sets D and F is called the **intersection** of the two sets, written $D \cap F$:

$D \cap F = \{3, 9, 15, 21\}$

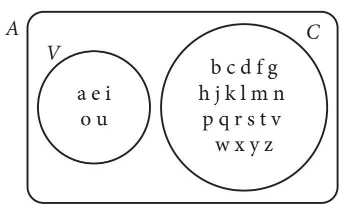
The intersection contains numbers that belong to both sets, i.e. the odd multiples of 3 between 1 and 25.

Now $n(D) = 13$, $n(F) = 8$, $n(D \cup F) = 17$, $n(D \cap F) = 4$. Hence:

$$n(D \cup F) = n(D) + n(F) - n(D \cap F)$$

Because sets D and F contain some common elements, the two sets are not mutually exclusive. Also, sets D and F are each **subsets** of set S, because all their elements are in set S.

Note that sets do not always overlap. If $A = \{\text{letters of the alphabet}\}$, $V = \{\text{vowels}\}$ and $C = \{\text{consonants}\}$, this information can be shown on a Venn diagram:



Sets V and C have no members in common, so they do not overlap. Hence $V \cap C = \{\}$, where $\{\}$ is called the empty set.

All members of set A are in either set V or set C. Because of this, sets V and C are called **mutually exclusive** or **disjoint**.

Now $n(V) = 5$, $n(C) = 21$, $n(V \cup C) = 26$, $n(V \cap C) = 0$.

Hence, for disjoint or mutually exclusive sets:

$$n(V \cup C) = n(V) + n(C)$$

Sets V and C are each subsets of set A, because all their elements are in set A. They are also called **complementary** sets, because together they make up set A with no overlapping.

Example 1

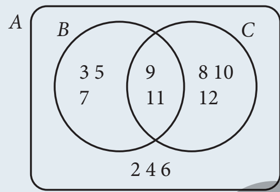
A is the set of possible sums from the numbers showing when two dice are rolled. B is the set of odd sums and C is the set of sums that are more than 7. Show this information on a Venn diagram and find:

(a) $B \cup C$ (b) $B \cap C$

Solution

List the elements in each set:

Venn diagram:



$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,

$B = \{3, 5, 7, 9, 11\}$, $C = \{8, 9, 10, 11, 12\}$

Elements common to B and C: $\{9, 11\}$

(a) $B \cup C = \{3, 5, 7, 8, 9, 10, 11, 12\}$

(b) $B \cap C = \{9, 11\}$

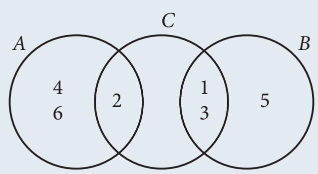
Example 2

A die is rolled on a flat surface. A is the set of even results, B is the set of odd results and C is the set of results less than 4. Show this information on a Venn diagram and find:

(a) $A \cup C$ (b) $C \cup B$ (c) $A \cup B \cup C$ (d) $A \cap B$ (e) Which two sets are mutually exclusive?

Solution

List the sets: $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{1, 2, 3\}$



There are 3 intersecting sets, so 3 groups needed.

Common elements between sets: 2 is common to A and C

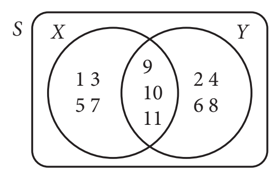
1, 3 are common to B and C

(a) $A \cup C = \{1, 2, 3, 4, 6\}$ (b) $C \cup B = \{1, 2, 3, 5\}$ (c) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$

(d) $A \cap B = \{\}$ This set is called the empty set as it does not contain any elements.

(e) Since the intersection of sets A and B is the empty set then these two sets are mutually exclusive.

EXERCISE 9.1 SETS AND SET NOTATION



1 For the Venn diagram shown, the set S is given by:

A $S = \{1, 3, 5, 7, 9, 10, 11\}$ B $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

C $S = \{2, 4, 6, 8, 9, 10, 11\}$ D $S = \{9, 10, 11\}$

To show the information given in questions **2** to **5**, draw a Venn diagram for each.

2 S is the set of possible sums from the numbers showing when two dice are rolled, D is the set of even sums, F is the set of sums less than 6. Find: (a) $D \cup F$ (b) $D \cap F$

3 A coin is tossed and a die is rolled. S is the set of all possible results obtained, H is the set of all results containing heads, G is the set of all results containing an odd number. Find:

(a) $H \cup G$ (b) $n(H \cap G)$

4 S is the set of all possible outcomes on a French roulette wheel, B is the set of black numbers, R is the set of red numbers. (For a description of a French roulette wheel, see Exercise 9.1 question **23**.) Find:

(a) $B \cup R$ **(b)** $B \cap R$ **(c)** $n(B \cup R)$ **(d)** $n(B \cap R)$ **(e)** $n(S)$

5 $S = \{1, 2, 3, \ldots 25\}$, $E = \{\text{even numbers in set } S\}$, $P = \{\text{perfect squares in set } S\}$. Find:

(a) $n(S)$ **(b)** $n(E)$ **(c)** $n(P)$ **(d)** $n(E \cap P)$ **(e)** $n(E \cup P)$

6 In a group of 25 students, 18 study French, 12 study German and 5 study neither French nor German. Find how many students:

(a) study only French **(b)** study only German **(c)** study only one language.

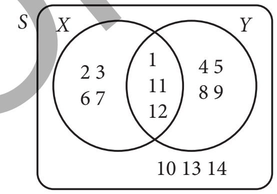
7 $S = \{1, 2, 3, \ldots 20\}$, $K = \{\text{multiples of 3 in set } S\}$, $L = \{\text{multiples of 6 in set } S\}$. Find:

(a) $K \cup L$ **(b)** $K \cap L$ **(c)** $n(K \cap L)$

8 A die is rolled on a flat surface. A is the set of even results, B is the set of odd results, C is the set of results more than 4. Find: **(a)** $A \cap B$ **(b)** $A \cup B$ **(c)** $B \cap C$ **(d)** $n(A \cap C)$

9 Two dice are rolled. Of the results, $E = \{\text{even sums}\}$, $F = \{\text{odd sums}\}$, $G = \{\text{sums where one die shows a 6}\}$. Find: **(a)** $n(E \cap F)$ **(b)** $n(F \cap G)$ **(c)** $n(E \cup F)$ **(d)** $n(E \cup G)$

10 For the Venn diagram shown, indicate whether each statement is correct or incorrect.



(a) $S = \{1, 2, 3, \ldots 12\}$ **(b)** $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12\}$
(c) $X \cap Y = \{1, 11, 12\}$ **(d)** $S = \{1, 2, 3, \ldots 14\}$

9.2 INTRODUCTION TO PROBABILITY

People often make statements like the following:

- It will **probably** rain today.
- I have a **good chance** of passing my exams.
- There is a **50-50 chance** that heads will come up when I toss a coin.
- I have a **slight chance** of being dux of the class.

In each case there is some doubt about the outcome, but the degree of doubt is different. **Probability** is about this doubt: it is the study of events that may or may not happen, rather than of events that will happen or that have already happened.

If you toss a coin, the result must be heads (H) or tails (T). There are only two possible outcomes, H or T, and each outcome is **equally likely**. You say that the probability of heads is $\frac{1}{2}$, and you can write this as:

$$P(\text{H}) = \frac{1}{2}$$

This probability be expressed in words in a variety of ways:

- There is a 50-50 chance that heads will turn up.
- There is a 1-in-2 chance that heads will turn up.
- The odds that heads will turn up are 1 to 1.
- It is an even-money bet that heads will turn up.

Before beginning a game of football, cricket, basketball etc., the team captains usually toss a coin. Why?

Equally likely events are events where each outcome has the same chance of occurring.

Coin experiment

Toss a coin 10 times, 50 times, 100 times, 200 times and 500 times, counting the number of heads tossed each time. (This can be simulated using a calculator or spreadsheet.) Complete the following table:

Number of coin tosses	10	50	100	200	500
Number of heads					
Relative frequency (number of heads / number of throws)					

If you toss the coin 10 times, how many heads would you expect? If you toss the coin 50 times, how many heads would you expect? Would you be surprised if you didn't get exactly the number that you expected?

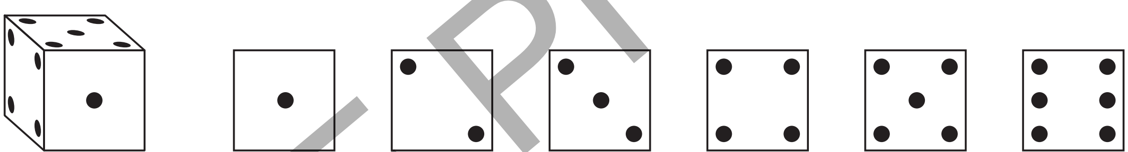
The relative frequency of each experiment gives you the actual probability of that outcome occurring. Is it as close to 0.5 as you expected? Did the relative frequency get closer to 0.5 as you increased the number of throws?

MAKING CONNECTIONS

Coin experiment

Use a spreadsheet to simulate the coin experiment. Select the number of coin tosses and investigate the results.

An American soldier, while a prisoner of war in World War II, passed the time by tossing a coin 1000 times. He performed this experiment ten times and obtained the following numbers of heads: 502, 511, 497, 529, 504, 476, 507, 528, 504, 529. Are these numbers surprising?



Die   Faces

A die (plural: dice) is a cube with six faces numbered 1 to 6, as shown above. If a die is rolled on a flat surface, one of the six numbers must appear on the top. (For example, the number 5 appears on top of the die in the picture above.)

- Each number from 1 to 6 is **equally likely** to turn up. Why?
- What is the probability that a 5 will appear on top?

There are six possible outcomes. Of these six outcomes, the outcome that you are interested in is a 5. This is called the **favourable** outcome. The probability of a 5 appearing is $\frac{1}{6}$, so the probability of a favourable outcome is written as is $P(5) = \frac{1}{6}$.

This can be expressed in a variety of ways:

- There is a 1-in-6 chance that a 5 will turn up.
- The odds are 5 to 1 against a 5 turning up. This means that for every 1 favourable outcome there are 5 unfavourable outcomes.

The probability of an outcome is defined as the ratio of the number of favourable outcomes to the number of possible outcomes, assuming that the outcomes are equally likely. Thus, the probability $P(A)$ of a particular result A is:

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \quad \textbf{OR} \quad P(\text{Event}) = \frac{\text{number of ways Event can occur}}{\text{number of possible outcomes}}$$

Dice experiment

Roll a die 6 times, 60 times, 120 times, 300 times and 600 times, counting the number of times a 5 (or another particular number) is rolled. (This can be simulated using a calculator or spreadsheet.) Complete the following table:

Number of rolls of the die	6	60	120	240	300	600
Number of times 5 appears						
Relative frequency (number of 5s / number of rolls)						

If you roll a die 6 times, how many 5s would you expect? If you roll a die 60 times, how many 5s would you expect? Would you be surprised if you didn't get exactly the number you expected?

The relative frequency of each experiment gives you the actual probability of that outcome occurring. Is it as close to $\frac{1}{6}$ as you expected? Did the relative frequency get closer to $\frac{1}{6}$ (≈ 0.1667) as you increased the number of throws?

MAKING CONNECTIONS

Dice experiment

Use a spreadsheet to simulate the dice-roll experiment. Select the number of rolls and investigate the results.

From the definition of probability as the number of favourable outcomes divided by the number of possible outcomes, you can see that if the number of favourable outcomes is equal to the number of possible outcomes then the probability is 1. If the probability is 1, then the event is **certain** to happen. For example:

- It is certain that the Sun will rise in the east tomorrow.
- The probability of obtaining a number less than 7 when a die is rolled $= \frac{6}{6} = 1$ (a certainty).

List some other situations where the probability is 1.

If there are no favourable outcomes, then the probability is 0. For example, many athletes can run very fast, but no athlete can run at the speed of light. The probability that an athlete will run faster than light is 0. Similarly, the probability that a 7 will appear when a normal die is rolled $= \frac{0}{6} = 0$.

Probabilities must always be from 0 to 1 inclusive:

- $0 \le P(\text{Event}) \le 1$
- If $P(\text{Event}) = 0$, the event is impossible.
- If $P(\text{Event}) = 1$, the event is certain.

To win Lotto®, you must pick six winning numbers out of the numbers 1 to 45. The number of different ways that you can choose 6 numbers from 45 numbers is 8 145 060, but only one of these ways is favourable (i.e. will win):

$P(\text{6 winning numbers}) = \frac{1}{8145060}$, $\approx 0.000\,000\,1$, which is rather small.

As another example, the following table shows the number of live births and the number of male and female babies born in Australia over the three-year period 2000 to 2002.

Year	Live births	Boys	Girls	Proportion of boys
2000	249 636	128 190	121 446	0.514
2001	246 394	126 298	120 096	0.513
2002	250 988	128 623	122 365	0.512

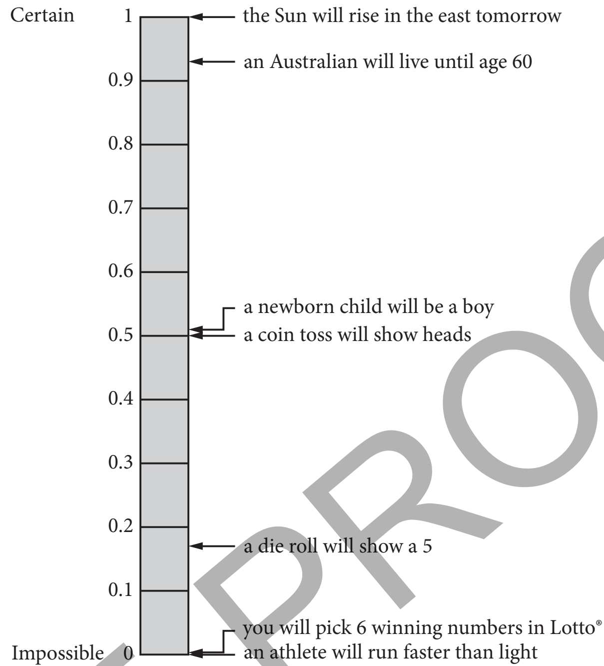
With rare exceptions, statistics over the years indicate a slight excess of male over female births.

$$P(\text{newborn is male}) = \frac{\text{number of boys born}}{\text{total number of births}} \approx 0.51$$

Insurance companies prepare 'life tables', which show the number of Australian people per 100 000 who are statistically expected to be alive at any given age. For example, a table like this could show that 92 515 people out of every 100 000 people born are expected to be still alive after 60 years:

$$P(\text{an Australian will live until age 60}) = \frac{92515}{100000} = 0.93$$

The following scale shows the probabilities of the events discussed.



Example 3

A die is rolled on a flat surface. What is the probability that the number appearing is:

(a) odd

(b) divisible by 3

(c) odd **or** divisible by 3

(d) odd **and** divisible by 3?

Solution

The six possible outcomes from rolling a die are equally likely.

(a) There are three odd numbers (1, 3, 5), so there are 3 favourable outcomes.

$$P(\text{odd}) = \frac{3}{6} = \frac{1}{2}$$

(b) There are two numbers divisible by 3 (3, 6), so there are 2 favourable outcomes.

$$P(\text{divisible by 3}) = \frac{2}{6} = \frac{1}{3}$$

(c) 1, 3, 5, 6 are favourable, so there are 4 favourable outcomes. Note carefully the word 'or' in the question.

$$P(\text{odd or divisible by 3}) = \frac{4}{6} = \frac{2}{3}$$

(d) 3 is the only number that is odd and also divisible by 3, so there is only 1 favourable outcome.

$$P(\text{odd and divisible by 3}) = \frac{1}{6}$$

Complementary events

The **complement of event** A is the event 'A **does not occur**', and can be denoted by $\overline{A}$.

If an experiment has n possible outcomes, m of which are associated with event A, then $(n - m)$ outcomes are associated with event $\overline{A}$, and so:

$$P(A) = \frac{m}{n} \qquad P\left(\overline{A}\right) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

Thus:

$$P(A) + P\left(\overline{A}\right) = 1$$

In general, P(Event does not occur) = 1 − P(Event occurs), or:

$$P\left(\overline{A}\right) = 1 - P(A)$$

Example 4

A die is rolled on a flat surface. What is the probability that the number rolled is **not** 4?

Solution

$P(4) = \frac{1}{6}$ $\qquad$ $P(A) = \frac{1}{6}$

$\therefore\ P(\text{not } 4) = 1 - \frac{1}{6} = \frac{5}{6}$ $\qquad$ $P\left(\overline{A}\right) = 1 - \frac{1}{6} = \frac{5}{6}$

A standard deck (or pack) of playing cards consists of 52 cards, made up of 4 sets of 13 cards in groups called a suit. The suits are called hearts, diamonds, spades and clubs and have symbols on them that represent the name of the suit. The hearts have red hearts on them, the diamonds have red diamonds on them, the spades have a black shape called a spade on them and the clubs have a black shape like a clover leaf on them.

This means that two of the suits have red coloured symbols on them and the other two of the suits have black coloured symbols on them. Hence half the deck is red cards and half the deck is black cards.

In each suit the cards are numbered 2, 3, 4, 5, 6, 7, 8, 9 ,10, J, Q, K, A, where J stands for Jack, Q stands for Queen, K stands for King and A stands for Ace.

The jacks, queens and kings are called court cards or picture cards.

Example 5

From a standard deck of 52 playing cards, one card is drawn at random. What is the probability that it is:

(a) a diamond **(b)** not a spade **(c)** an ace

(d) a diamond **or** an ace **(e)** a diamond **and** an ace **(f)** a heart **and** the ace of clubs?

Solution

There are 52 possible outcomes.

(a) There are 13 diamonds: $P(\text{diamond}) = \frac{13}{52} = \frac{1}{4}$

(b) There are 13 spades, so there are 39 cards that are not spades: $P(\text{not spade}) = \frac{39}{52} = \frac{3}{4}$

Or, using complementary events: $P(\text{not spade}) = 1 - P(\text{spade}) = 1 - \frac{1}{4} = \frac{3}{4}$

(c) There are 4 aces: $P(\text{ace}) = \frac{4}{52} = \frac{1}{13}$

(d) There are 13 diamonds and 4 aces, but one ace is also a diamond, so there are $13 + 3 = 16$ favourable outcomes: $P(\text{diamond } \textbf{or} \text{ ace}) = \frac{16}{52} = \frac{4}{13}$

(e) The ace of diamonds is the only card that is a diamond and also an ace: $P(\text{diamond } \textbf{and} \text{ ace}) = \frac{1}{52}$

(f) There is no card that is a heart and also the ace of clubs: $P(\text{heart } \textbf{and} \text{ ace of clubs}) = 0$

EXERCISE 9.2 INTRODUCTION TO PROBABILITY

1 Select 20 lines of text from a page in a book. Count the number of times each letter of the alphabet appears. Would you expect each of the 26 letters of the alphabet to appear approximately the same number of times? If a letter in your 20 lines is selected at random, would you say that the probability that the letter is 'e' is the same as the probability that it is 'z'?

2 Select a page in a telephone directory (or search at random on a telephone directory website) and count the number of times that each digit 0, 1, 2, 3, … 9 appears as the last digit of a phone number. Prepare a table showing your results.

(a) Did you expect results like these?

(b) Would similar results have occurred if the first digit of each number was recorded? Why?

(c) What would have happened if the first letter of each person's name was recorded?

3 What is wrong with the following arguments?

(a) The probability that a child born in Australia is born in New South Wales is $\frac{1}{7}$.

(b) Because there are fewer road accidents in Hobart than in Sydney, it is safer to drive in Hobart.

4 Which statement has a probability closest to 1?

A The cost of buying a house will increase this year.

B You will win the lottery.

C A person will be born in Sydney in the next hour.

D It will rain in Sydney tomorrow.

5 Perform the following experiment:
From a bag containing 7 black marbles and 3 white marbles, withdraw a marble, note its colour and then replace it in the bag. Shake the marbles and then withdraw a marble again. Repeat this 10, 50, 100, 150 and 200 times to complete the following table.

Number of withdrawals	10	50	100	150	200
Number of black marbles withdrawn					
Relative frequency (number of black / number of withdrawals)					

How many black marbles do you expect to get in each case?

Did the relative frequency get closer to 0.7 as you increased the number of withdrawals?

6 Perform the following experiment:
From a standard deck of 52 playing cards, withdraw a card and note whether it is a diamond, heart, spade or club. Replace the card in the deck, shuffle the cards and then withdraw another card. Repeat this 20, 40, 60, 80 and 100 times to complete the following table.

Number of withdrawals	20	40	60	80	100
Number of hearts withdrawn					
Relative frequency (number of hearts / number of withdrawals)					

How many hearts do you expect to get in each case?
Did the relative frequency get closer to 0.25 as you increased the number of withdrawals?

7 A box contains 50 matches, of which 40 are 'live' (ready to be struck) and the remainder 'dead' (used). A match is selected at random. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{live}) = \frac{4}{5}$ (b) $P(\text{dead}) = \frac{4}{5}$ (c) $P(\text{live}) = \frac{1}{5}$ (d) $P(\text{dead}) = \frac{1}{5}$

8 A carton contains a dozen eggs, of which three are brown and the rest are white. An egg is chosen at random from the carton. What is the probability that it is:

(a) brown (b) white?

9 A bag contains 2 white marbles, 2 black marbles and 1 red marble. A marble is selected from the bag. What is the probability that it is:

(a) red (b) black (c) not white
(d) red or black (e) not black, nor white (f) red or white?

10 A set of 20 cards is numbered 1, 2, 3, … 20. A card is drawn at random from the set. What is the probability that the number on the card is divisible by:

(a) 3 (b) 5 (c) 3 or 5 or both (d) 3 and 5?

11 A standard six-sided die is rolled on a flat surface. What is the probability that the number appearing is:

(a) greater than 2 (b) greater than 3 but less than 5
(c) odd (d) odd and divisible by 3?

12 A set of 10 cards is numbered 3, 4, 5, … 12. A card is drawn at random. What is the probability that the number on the card is:

(a) even (b) odd (c) odd or even (d) greater than 7 (e) divisible by 3
(f) even and divisible by 3 (g) divisible by 5 (h) odd or divisible by 5, but not both?

13 A letter is chosen at random from the letters of the word SUNDAY. What is the probability that it is:

(a) a vowel (b) a consonant (c) D or A?

14 A card is drawn at random from a standard deck of 52 playing cards. What is the probability that it is:

(a) a heart (b) a king (c) a heart or a king
(d) a heart **and** a king (e) the queen of diamonds?

15 A number is chosen at random from the numbers 7, 8, 9, … 15. What is the probability that the number is:

(a) odd (b) odd and divisible by 5 (c) odd but not divisible by 5
(d) a multiple of 2 (e) a multiple of 3 (f) a multiple of both 2 and 3
(g) a multiple of 2 or 3 (h) a multiple of 2 or 3, but not both?

16 A bag contains 5 blue marbles, 3 red marbles and 2 green marbles. One marble is drawn at random from the bag. What is the probability that it is:

(a) blue (b) red (c) not green (d) blue or green (e) not red, nor blue?

17 A box contains marbles of the same size but different colours: red, white and blue. If a marble is drawn at random, the probability that it is red is the same as the probability that it is white and twice the probability that it is blue.

(a) What is the smallest number of marbles that the box could have?

(b) If a marble is chosen at random, what is the probability that it is:

(i) red **(ii)** white **(iii)** red or white **(iv)** not blue?

18 A bag contains 10 balls, of which 6 are red, 3 are yellow and 1 is white. A ball is drawn at random. What is the probability that it is:

(a) red **(b)** not yellow **(c)** white or yellow **(d)** neither red nor white **(e)** not red?

19 A bag contains 12 discs, of which 7 are black, 3 are white and 2 are red. A black disc is withdrawn. What is the probability that the next disc withdrawn is:

(a) black **(b)** not white **(c)** neither black nor white?

20 A survey of a certain district showed that 6% of families there have 1 child, 38% have 2 children, 42% have 3 children, and 10% have more than 3 children. A family from the district is selected at random. What is the probability that the family will have:

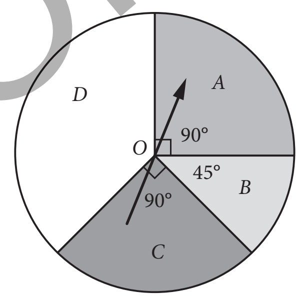
(a) some children **(b)** no children

(c) at least 2 children **(d)** not more than 2 children?

21 The diagram shows a circular disc divided into four sections A, B, C and D. A pointer is free to spin, pivoted at the centre O. If the pointer is spun, find the probability that the end of the pointer stops inside:

(a) A **(b)** B **(c)** A or B

(d) not C **(e)** B, C or D

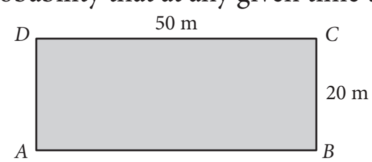


22 A horse is grazing inside an enclosed rectangular paddock 50 m by 20 m and is free to move anywhere inside the paddock. Assuming that the horse's position is random, what is the probability that at any given time the horse is:

(a) more than 5 m from the fence

(b) less than 5 m from the fence

(c) not more than 5 m from a corner?

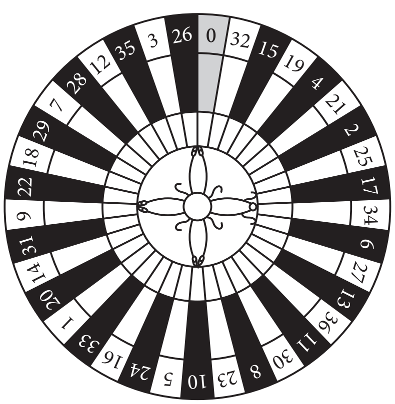


23 A French roulette wheel has 37 compartments around its rim. One of these is numbered 0 and is coloured green (shown as grey at right). The others are numbered 1 to 36, with half of these 36 compartments coloured red (shown as white at right) and the other half coloured black. The wheel is spun in one direction and a small ball is rolled in the opposite direction. The chances of the ball falling into any of the 37 compartments are equally likely. Find the probability that the ball will land on:

(a) black **(b)** an odd number

(c) a multiple of 3 **(d)** any number from 1 to 12

(e) 15



9.3 FINITE SAMPLE SPACES

A set of favourable outcomes can be considered an **event**. The rolling of even numbers with a die (3 favourable outcomes), the drawing of a heart from a standard deck of cards (13 favourable outcomes) or the tossing of heads with a coin (1 favourable outcome) are all examples of an event.

The rolling of a die, the drawing of cards from a deck, the tossing of a coin—that is, activities that will produce various events—are called **experiments** or **trials**.

In modern probability theory, all possible outcomes of an experiment are considered as points in a space, called a **sample space** S. If S contains a finite number of points n and if the outcomes of an experiment are equally likely, then you can assign each point (called a **sample point**) a probability of $\frac{1}{n}$. The sum of the probabilities of all the sample points is therefore 1.

If S is the sample space and A is a set of points drawn from the sample space (i.e. A is an event), then:

$$P(A) = \frac{\text{number of sample points in } A}{\text{number of sample points in } S} = \frac{n(A)}{n(S)} = \frac{|A|}{|S|}$$
$$= \frac{\text{number of outcomes corresponding to } A}{\text{total number of possible outcomes}}$$

If $S = \{a, b, c, \ldots x, y, z\}$ then $n(S) = 26$.

Let $A = \{a, e, i, o, u\}$ so $n(A) = 5$: $P(A) = \frac{n(A)}{n(S)} = \frac{5}{26}$

Thus the probability of selecting a vowel from S is $\frac{5}{26}$.

Mutually exclusive events

If two or more events cannot occur simultaneously then the events are said to be **mutually exclusive** or **disjoint**. In the language of the sample space, the events have no points in common.

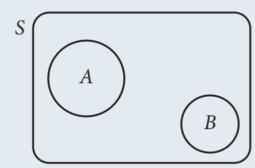
If a coin is tossed, either heads or tails may turn up, but both events cannot occur at the same time. If a card is drawn at random from a deck, it may be a heart or the ace of spades or some other card, but if the card is a heart then this excludes the possibility of it being the ace of spades. The two events 'heart' and 'ace of spades' are mutually exclusive. However, the events 'heart' and 'ace' are not mutually exclusive. Why?

Example 6

If one card is drawn at random from a standard deck of 52 playing cards, what is the probability that it is a heart or the ace of spades?

Solution

These two events are mutually exclusive, because they can't happen together.



$S = \{\text{playing cards}\}$ $\quad n(S) = 52$

$A = \{\text{heart}\}$ $\quad n(A) = 13$

$B = \{\text{ace of spades}\}$ $\quad n(B) = 1$

$A \cup B = A \text{ or } B = \{\text{heart or ace of spades}\}$

$n(A \text{ or } B) = 14$

$P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)} = \frac{14}{52} = \frac{7}{26}$

A and B are disjoint sets, so they don't intersect.

Now $P(A) = \frac{13}{52}$, $P(B) = \frac{1}{52}$ and so $P(A) + P(B) = \frac{13}{52} + \frac{1}{52} = \frac{14}{52} = \frac{7}{26}$

Because the sets A and B have no points in common and thus are mutually exclusive, then:

$P(A \text{ or } B) = P(A) + P(B)$

Example 7

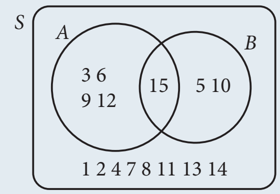
From a set of 15 cards numbered 1 to 15, one card is drawn at random. What is the probability that it is a multiple of 3 or 5 or both?

Solution

$S = \{1, 2, 3, \ldots 15\}$ $\quad n(S) = 15$

$A = \{3, 6, 9, 12, 15\}$ $\quad n(A) = 5$

$B = \{5, 10, 15\}$ $\quad n(B) = 3$



One card, 15, is common to both sets A and B:

$A \cap B = A \text{ and } B = \{15\}$ $\quad n(A \text{ and } B) = 1$

$A \cup B = A \text{ or } B = \{3, 5, 6, 9, 10, 12, 15\}$ $\quad n(A \text{ or } B) = 7$

$$\therefore P(A \text{ or } B) = \frac{n(A \text{ or } B)}{n(S)} = \frac{7}{15}$$

The notation 'A and B' means the elements common to both sets (in Example **7** above, the multiples of 3 and 5). This is also known as the intersection of the two sets.

The notation 'A or B' means the elements in either set (in the example above, the multiples of 3 or 5 or both). This is also known as the union of the two sets.

You have $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$, because the sets are not mutually exclusive. Thus:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This formula also works for mutually exclusive sets, where $P(A \text{ and } B) = 0$ and so the result becomes:

$$P(A \text{ or } B) = P(A) + P(B)$$
$$P(A \cup B) = P(A) + P(B)$$

Example 8

A card is drawn at random from a standard deck of 52 playing cards. R is the event 'drawing a red card,' H is the event 'drawing the ace of hearts,' A is the event 'drawing an ace.' Find:

(a) $\bar{R}$ **(b)** $\bar{A}$ **(c)** $P(R)$ **(d)** $P(\bar{R})$

(e) $P(R \text{ or } A)$ **(f)** $P(R \text{ or } H)$ **(g)** $P(R \text{ and } A)$ **(h)** $P\left(\bar{R} \text{ or } H\right)$

Solution

(a) $\bar{R} = \{\text{not red}\} = \{\text{black}\}$ **(b)** $\bar{A} = \{\text{not an ace}\}$ **(c)** $n(R) = 26, \quad P(R) = \frac{26}{52} = \frac{1}{2}$

(d) $n(\bar{R}) = 26, \quad P(\bar{R}) = \frac{26}{52} = \frac{1}{2}$ **OR** $P(\bar{R}) = 1 - P(R) = 1 - \frac{1}{2} = \frac{1}{2}$

(e) $n(R \cup A) = 26 + 4 - 2 = 28, \quad P(R \text{ or } A) = \frac{n(R \cup A)}{52} = \frac{28}{52} = \frac{7}{13}$

(f) $n(R \cup H) = n(R) = 26$ because the ace of hearts is a red card.

$$P(R \text{ or } H) = \frac{n(R \cup H)}{52} = \frac{26}{52} = \frac{1}{2}$$

(g) $n(R \cap A) = 2$ because there are 2 red aces.

$$P(R \text{ and } A) = \frac{n(R \cap A)}{52} = \frac{2}{52} = \frac{1}{26}$$

(h) $\bar{R} \cup H = \{\text{black or ace of hearts}\}, P(\bar{R} \text{ or } H) = \frac{n(\bar{R} \cup H)}{52} = \frac{27}{52}$

EXERCISE 9.3 FINITE SAMPLE SPACES

1 A card is drawn at random from a standard deck of 52 playing cards. A is the event 'drawing a heart', B the event 'drawing the ace of clubs', C is the event 'drawing an ace'. Find the following:

(a) $P(A)$ (b) $P(B)$ (c) $P(C)$ (d) $P(A \text{ or } B)$ (e) $P(A \text{ or } C)$
(f) $P(B \text{ or } C)$ (g) Which pairs of events A and B, A and C, or B and C are mutually exclusive?

2 From a set of 17 cards numbered 1, 2, 3, … 17, one card is selected at random. A is the event 'a multiple of 3', B is the event 'a multiple of 8', C is the event 'a multiple of 5'. Find the following:

(a) $P(A)$ (b) $P(B)$ (c) $P(C)$ (d) $P(\overline{A})$ (e) $P(\overline{B})$
(f) $P(A \text{ or } B)$ (g) $P(A \text{ or } C)$ (h) $P(\overline{A} \text{ or } B)$ (i) $P(A \text{ or } \overline{C})$ (j) $P(\overline{B} \text{ or } \overline{C})$
(k) Which pairs of the events A, B and C are mutually exclusive?

3 A standard die is rolled on a flat surface. A is the event 'an even number', B is the event 'an odd number', C is the event 'a multiple of 5'. Which of the following statements is true?

A $P(A \text{ or } B) = 0.5$ **B** $P(\overline{A} \text{ or } B) = 1$ **C** $P(A \text{ or } C) = 0.5$ **D** $P(\overline{B} \text{ or } \overline{C}) = \frac{5}{6}$

4 A container holds 8 red marbles, 7 white marbles and 5 black marbles. One marble is drawn at random from the container. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{red or black}) = \frac{3}{4}$ (b) $P(\text{not white}) = \frac{1}{4}$
(c) $P(\text{neither black nor white}) = \frac{2}{5}$ (d) $P(\text{not red}) = \frac{3}{5}$

5 A coin is tossed three times. A is the event 'at least two heads', B is the event 'head, tail, head', C is the event 'not more than one head'. Find:

(a) $P(A)$ (b) $P(B)$ (c) $P(C)$ (d) $P(A \text{ or } B)$ (e) $P(A \text{ or } C)$ (f) $P(B \text{ or } C)$
(g) $P(A \text{ or } B \text{ or } C)$ (h) Which pairs of the events A, B and C are mutually exclusive?

6 An integer is chosen at random from the first 50 positive integers. A is the event 'divisible by 2', B is the event 'divisible by 3', C is the event 'divisible by 5'. Find:

(a) $P(A \text{ and } B)$ (b) $P(A \text{ and } C)$ (c) $P(B \text{ and } C)$ (d) $P(A \text{ and } B \text{ and } C)$
(e) $P(A \text{ or } B)$ (f) $P(A \text{ or } C)$ (g) $P(B \text{ or } C)$ (h) $P(A \text{ or } B \text{ or } C)$

7 A die is rolled twice and the numbers rolled are added together. A is the event 'the sum is 4 or more', B is the event 'the sum is less than 6', C is the event 'the two numbers rolled are the same'. Find:

(a) $P(A \text{ and } B)$ (b) $P(A \text{ and } C)$ (c) $P(B \text{ and } C)$ (d) $P(A \text{ and } B \text{ and } C)$
(e) $P(A \text{ or } B)$ (f) $P(A \text{ or } C)$ (g) $P(B \text{ or } C)$ (h) $P(A \text{ or } B \text{ or } C)$

8 A die is rolled twice and the numbers rolled are added together. A is the event 'the sum is greater than 5', B is the event 'the sum is less than 8'. Find:

(a) $P(A)$ (b) $P(B)$ (c) $P(A \text{ and } B)$ (d) $P(A \text{ or } B)$.
(e) Hence show that $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. Are A and B mutually exclusive?

9 If $P(A) + P(B) = 1$, does it follow that one of these events must occur?

10 $P(A \text{ or } B) = \frac{15}{16}$, $P(A \text{ and } B) = \frac{1}{8}$, $P(A) = \frac{3}{8}$. Find $P(B)$.

11 In a group of 50 students, 30 study Mathematics, 25 study Physics and 20 study both Mathematics and Physics. One student is selected at random from the group. What is the probability that this student studies:

(a) Mathematics but not Physics (b) Physics but not Mathematics
(c) neither Physics nor Mathematics?

12 From a group of 100 students, 50 study History, 30 study English and 20 study both. If a student is selected at random from the group, what is the probability that the student studies:

(a) at least one of these subjects (b) History but not English
(c) History, given that the student also studies English?

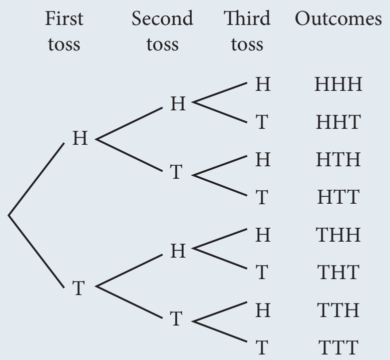
9.4 SUCCESSIVE OUTCOMES

It is important to be able to organise information so that probabilities can be calculated easily, especially when there are multiple events occurring successively. A **tree diagram** is one of the most common ways to do this.

Example 9

A coin is tossed three times. Use a tree diagram to show the possible outcomes.

Solution



Each time a coin is tossed, it can land as either a head (H) or a tail (T).

There are 8 possible outcomes ($2 \times 2 \times 2 = 2^3$) for this experiment. How you describe the outcomes depends on whether or not the order of the events is important.

If the order is important, then the outcomes HHT, HTH and THH are three different outcomes. If order is not important, then these three results represent the same single outcome: 2 heads and 1 tail.

When order is not important, there are only 4 possible outcomes for this experiment: 3 heads, 2 heads and 1 tail, 1 head and 2 tails, or 3 tails.

As well as listing outcomes, a tree diagram can be used to show the probability of each outcome in a repeated experiment. The fundamental counting principle (or 'multiplication principle'—see below) means that you can multiply the successive probabilities along the branches to obtain the probability of a final outcome.

Example 10

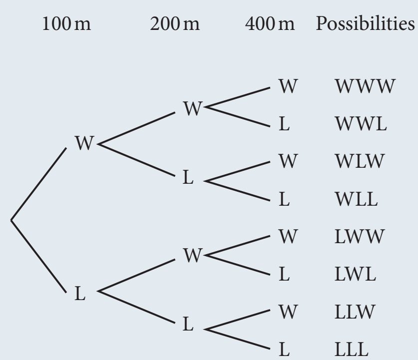
At a school sports day Joely runs in the 100 m, 200 m and 400 m races. In terms of win (W) and loss (L), there are two possible outcomes for each race.

(a) How many possibilities are there for the three races?

(b) What are the probabilities of Joely winning 3, 2, 1 or no races?

Solution

(a) There are 8 possible outcomes ($2 \times 2 \times 2$). WWW represents a win in each of the three races; WWL, for example, represents a win in 100 m and 200 m and a loss in the 400 m.



(b) The question does not say that Joely has an equally likely chance of winning each race, so the question cannot be answered due to lack of information.

However, you can find the probabilities if you assume that she has a 50-50 chance of winning each race, in which case:

$P(\text{wins 3 races}) = \frac{1}{8}$ $\quad P(\text{wins 2 races}) = \frac{3}{8}$

$P(\text{wins 1 race}) = \frac{3}{8}$ $\quad P(\text{wins 0 races}) = \frac{1}{8}$

Multiplication principle

Tree diagrams can be used to calculate probabilities. To obtain the probability of a final outcome, you can multiply the probabilities of the successive intermediate outcomes along the branches. For example:

> Number of ways a two-part outcome can occur $= m \times n$
> (where m = number of ways the first part can occur,
> n = number of ways the second part can occur)

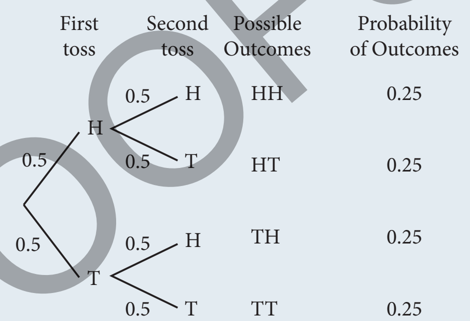
Example 11

A coin is tossed twice. Draw a tree diagram to show the possible outcomes, marking the probability of each outcome on each branch. Use your tree diagram to find:

(a) $P(2 \text{ heads})$ **(b)** $P(1 \text{ head})$ **(c)** $P(0 \text{ heads})$

Solution

$P(\text{H}) = P(\text{T}) = 0.5$



Find each final probability by multiplying the successive probabilities along each branch of the tree diagram.

(a) $P(2 \text{ heads}) = P(\text{H}) \times P(\text{H}) = 0.5 \times 0.5 = 0.25$

(b) There are 2 branches that have the outcome of 1 head, so their probabilities must be added together.

$$\begin{aligned} P(1 \text{ head}) &= P(\text{H}) \times P(\text{T}) + P(\text{T}) \times P(\text{H}) \\ &= 0.5 \times 0.5 + 0.5 \times 0.5 \\ &= 0.5 \end{aligned}$$

(c) $P(0 \text{ heads}) = P(\text{T}) \times P(\text{T}) = 0.5 \times 0.5 = 0.25$

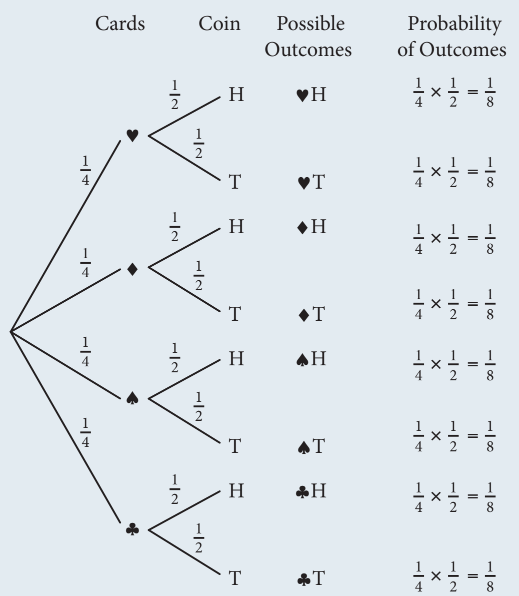
Example 12

A card is drawn from a standard deck of 52 playing cards and a coin is tossed. In this example, you are only concerned with the suit of the card (i.e. whether hearts, diamonds, clubs or spades). Draw a tree diagram to show the possible outcomes, marking the probability of each outcome on each branch. Use your tree diagram to find:

(a) $P(\text{spade and heads})$ **(b)** $P(\text{red card and tails})$ **(c)** $P(\text{not a club and heads})$

Solution

Multiply successive probabilities along the branches.



(a) $P(\text{spade and heads}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

(b) $P(\text{red card and tails}) = \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} = \frac{1}{4}$

OR, using successive probabilities:

$P(\text{red card}) = \frac{1}{2}$, $P(\text{tails}) = \frac{1}{2}$

$P(\text{red card and tails}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(c) $P(\text{not a club and heads}) = 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$

EXERCISE 9.4 SUCCESSIVE OUTCOMES

1 A coin is tossed and a die is rolled. Draw a tree diagram to show the outcomes, marking the probability of the outcome on each branch. If each outcome is equally likely, find the probability of:

(a) a head and an even number (b) a tail and a number greater than 4.

2 Two cubes each have two faces painted red, another two faces painted white and the remaining two faces painted blue. Both cubes are rolled on a table like dice. Draw a tree diagram to show the possible outcomes of the colours that are rolled. If each outcome is equally likely, what is the probability that:

(a) both colours rolled are the same (b) one colour rolled is red and the other is white?

3 A committee consisting of one boy and one girl is to be selected from three girls (Anna, Kate and Sue-Lin) and four boys (Brent, Jimmy, Mustafa and Rajiv). What is the probability that the committee consists of Kate and Rajiv?

4 A restaurant has a set menu that offers a choice of three different entrées, three different main courses and three different desserts. The entrées are vegetable soup, spring rolls and calamari; the main courses are roast beef, roast eggplant and spaghetti bolognese; the desserts are cheesecake, ice-cream and fruit salad. Draw a tree diagram to show all the menu choices that include one item from each course.

(a) If you must choose one item from each course, how many different meal choices are there?

If each choice is equally likely, find the probability of selecting:

(b) vegetable soup, roast beef, and either cheesecake or ice-cream

(c) spring rolls, not roast beef, and fruit salad.

5 A die is rolled twice and the numbers rolled are added together. Which of the following events has a probability of $\frac{1}{18}$?

A both numbers rolled are the same B the sum is 5 C the sum is 3 D the sum is 12

6 A coin is tossed three times. Illustrate on a tree diagram the eight possible outcomes. Use the tree diagram to find the probability of tossing:

(a) 3 tails (b) 1 head (c) 2 heads (d) 3 heads

7 Three students, Kristian, Soung-ho and Amélie, work independently on a mathematics problem. Each student has a 50-50 chance of success. Draw a tree diagram and use it to indicate whether each of the following statements is correct or incorrect.

(a) $P(\text{all successful}) = \frac{1}{2}$ (b) $P(\text{only Kristian and Amélie successful}) = \frac{1}{8}$

(c) $P(\text{none successful}) = \frac{3}{8}$ (d) $P(\text{at least one is successful}) = \frac{7}{8}$

8 One cube has 4 red faces and 2 white faces, another has 3 red and 3 white faces, and a third cube has 2 red and 4 white faces. The three cubes are rolled like dice.

(a) Draw a tree diagram to show all possible outcomes and the associated probabilities.

What is the probability that:

(b) 3 red faces are rolled

(c) only one cube rolls a white face

(d) more red faces than white faces are rolled?

9 A 50-cent coin, a 20-cent coin and a 10-cent coin are tossed. What is the probability of:

(a) 3 heads (b) a head only with the 50-cent coin (c) at least 2 tails?

10 A card is drawn at random from a standard deck of 52 playing cards. It is replaced, the deck is shuffled and a card is drawn again. What is the probability that:

(a) both cards are spades (b) neither card is a heart (c) both cards are the same suit?

9.5 INDEPENDENT EVENTS

Two events are **independent** if the outcome of each event is not affected by the outcome of the other event. This means that the probability of both events together is the same as the probability of both events separately, and so their probabilities can be multiplied together as if they are separate successive outcomes (as shown in the previous section).

In other words:

If events A and B are independent, then: $P(A \text{ and } B) = P(A) \times P(B)$
Or, writing $P(A$ and $B)$ as the 'product' of A and B: $P(AB) = P(A) \times P(B)$
$P(A \cap B) = P(A) \times P(B)$

Many of the questions in Exercise 9.4 involved independent events.

Example 13

If two dice are rolled, what is the probability of rolling an even number with the first die and a 3 or a 5 with the second die?

Solution

The sample space consists of $6 \times 6 = 36$ points, as listed below. Each sample point is represented for convenience below as (**a**, b), where **a** is the number on the first die and b is the number on the second die:

(**1**, 1)	(**2**, 1)	(**3**, 1)	(**4**, 1)	(**5**, 1)	(**6**, 1)
(**1**, 2)	(**2**, 2)	(**3**, 2)	(**4**, 2)	(**5**, 2)	(**6**, 2)
(**1**, 3)	(**2**, 3)	(**3**, 3)	(**4**, 3)	(**5**, 3)	(**6**, 3)
(**1**, 4)	(**2**, 4)	(**3**, 4)	(**4**, 4)	(**5**, 4)	(**6**, 4)
(**1**, 5)	(**2**, 5)	(**3**, 5)	(**4**, 5)	(**5**, 5)	(**6**, 5)
(**1**, 6)	(**2**, 6)	(**3**, 6)	(**4**, 6)	(**5**, 6)	(**6**, 6)

There are 18 points that correspond to event A, an even number with the first die and any number with the second die. There are 12 points that correspond to event B, any number with the first die and a 3 or a 5 with the second die.

Thus: $P(A) = \frac{18}{36} = \frac{1}{2}$ and $P(B) = \frac{12}{36} = \frac{1}{3}$.

The intersection of A and B is the set of 6 points above that are shaded.

$A \cap B = \{(2, 3), (2, 5), (4, 3), (4, 5), (6, 3), (6, 5)\}$

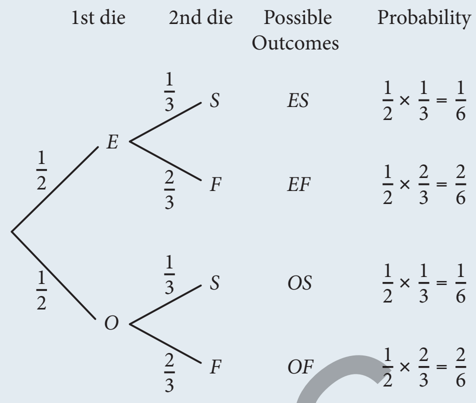
Hence: $P(A \text{ and } B) = \frac{6}{36} = \frac{1}{6}$

And: $P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \text{ and } B)$

This result conforms to our definition of independent events, as well as to the everyday meaning of the word 'independent'. Clearly, whichever number the first die happens to roll will have absolutely no influence on the second die's roll.

Alternatively:
This example could also be solved by a careful observation of the 36 sample points. Observe that $\frac{1}{2}$ of the outcomes contain an even number with the first die; then note that $\frac{1}{3}$ of this $\frac{1}{2}$ (that is, $\frac{1}{6}$) contain an even number with the first die **and** a 3 or a 5 with the second die.

We can think of rolling two dice as a two-stage process, which can be represented by a tree diagram with two sets of branches. For convenience, we label an even number on the first die E, an odd number on the first die O, a 3 or a 5 with the second die a success S, and 'not a 3 or a 5' a failure F.



The tree diagram shows the four possible outcomes, but they are not all equally likely. An even number E with the first die and a 3 or a 5 with the second die is given by ES with probability $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$.

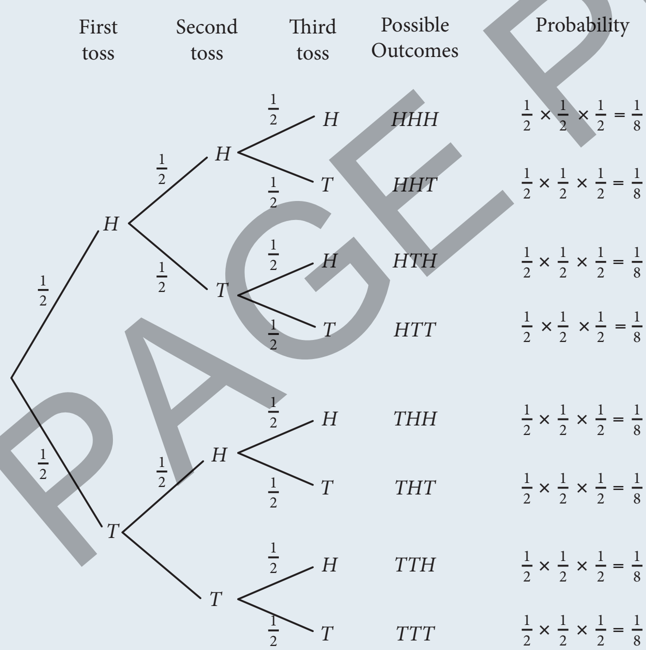
Example 14

A fair coin is tossed 3 times. What is the probability of:

(a) 2 heads and 1 tail, in any order (b) at least 1 head?

Solution

The experiment of tossing a coin 3 times can be represented by a tree diagram with 3 sets of branches. As usual, H = head and T = tail:



Hence $S = \{(HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT)\}$

$\therefore\ n(S) = 8$ and each of the 8 outcomes is equally likely.

(a) From observation of the 8 sample points, we can pick out the points corresponding to the event A, '2 heads and 1 tail': $A = \{(HHT), (HTH), (THH)\}$

$$\therefore\ P(\text{2 heads and 1 tail}) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

(b) Each of the 8 outcomes except (TTT) has at least 1 head: $\therefore\ P(\text{at least 1 head}) = 1 - P(\text{0 heads})$

$$= 1-\frac{1}{8}=\frac{7}{8}$$

Example 15

Two friends, Suzy and Dimitri, often play golf and tennis with each other. Overall, Suzy tends to win 3 rounds of golf out of every 5 and 1 game of tennis out of every 4. If Suzy and Dimitri play one round of golf and one game of tennis, find the probability that Suzy:

(a) wins both

(b) loses both

(c) wins only the round of golf

(d) wins either the golf or the tennis but not both.

Solution

Assume that no game ends in a draw.

Let A be the event 'Suzy wins golf', so $\bar{A}$ is the event 'Suzy loses golf'. From the overall probabilities:

$$P(A)=\frac{3}{5} \text{ and } P(\bar{A})=\frac{2}{5}$$

Let B be the event 'Suzy wins tennis', so $\bar{B}$ is the event 'Suzy loses tennis'.

$$P(B)=\frac{1}{4} \text{ and } P(\bar{B})=\frac{3}{4}$$

Assuming that A and B are independent events (which seems reasonable), then:

(a) $$\begin{aligned} P(A \text{ and } B) &= P(A)\times P(B) \\ &= \frac{3}{5}\times\frac{1}{4} \\ &= \frac{3}{20} \end{aligned}$$

(b) $$\begin{aligned} P(\bar{A} \text{ and } \bar{B}) &= P(\bar{A})\times P(\bar{B}) \\ &= \frac{2}{5}\times\frac{3}{4} \\ &= \frac{3}{10} \end{aligned}$$

(c) $$\begin{aligned} P(A \text{ and } \bar{B}) &= P(A)\times P(\bar{B}) \\ &= \frac{3}{5}\times\frac{3}{4} \\ &= \frac{9}{20} \end{aligned}$$

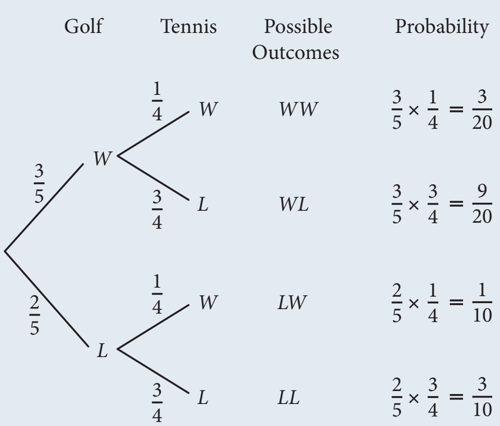
(d) For this outcome, Suzy either **(i)** wins golf and loses tennis, or **(ii)** loses golf and wins tennis.

(i) $$\begin{aligned} P(A \text{ and } \bar{B}) &= P(A)\times P(\bar{B}) \\ &= \frac{3}{5}\times\frac{3}{4} \\ &= \frac{9}{20} \end{aligned}$$

(ii) $$\begin{aligned} P(\bar{A} \text{ and } B) &= P(\bar{A})\times P(B) \\ &= \frac{2}{5}\times\frac{1}{4} \\ &= \frac{2}{20} \end{aligned}$$

(i) and **(ii)** are mutually exclusive, so $P\left((A \text{ and } \bar{B}) \text{ or } (\bar{A} \text{ and } B)\right)=\frac{9}{20}+\frac{2}{20}=\frac{11}{20}$.

This problem can also be solved using the tree diagram method. There are four possible outcomes (not all equally likely); if we denote a win by Suzy as W and a loss as L, then the possible outcomes are WW, WL, LW and LL. Their respective probabilities can be illustrated by the following tree diagram.



Example 16

A die is rolled, a coin is tossed and a letter is taken from the set {a, b, c}. Find the probability of each event described below.

(a) Event A: roll 3 with the die
(b) Event B: a head with the coin
(c) Event C: 'b' from the set {a, b, c}
(d) A and B and C.

Solution

Events A, B and C are independent.

(a) $P(A)=\frac{1}{6}$
(b) $P(B)=\frac{1}{2}$
(c) $P(C)=\frac{1}{3}$
(d) $P(A \text{ and } B \text{ and } C)=P(A)\times P(B)\times P(C)$

$$=\frac{1}{6}\times\frac{1}{2}\times\frac{1}{3}$$
$$=\frac{1}{36}$$

When finding the probability of an event, given that you know another event has occurred, the notation | is used to mean 'given'. Hence P(A|B) means 'the probability of event A, given the situation that event B has occurred'.

It is important to realise that:

> If events A and B are independent, then $P(A \mid B)=P(A)$ and $P(B \mid A)=P(B)$.

EXERCISE 9.5 INDEPENDENT EVENTS

1 You toss a coin and roll a die. What is the probability of:

(a) a head with the coin
(b) a number greater than 4 with the die
(c) a head and a number greater than 4?

2 In a class of 25 boys and 15 girls, 8 boys and 7 girls wear glasses. One student from the class is selected. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{boy})=\frac{5}{8}$
(b) $P(\text{wears glasses})=\frac{3}{8}$
(c) $P(\text{boy and wears glasses})=\frac{1}{5}$
(d) $P(\text{girl and does not wear glasses})=\frac{7}{40}$

3 A coin is tossed and a die is rolled. The probability of 'a head and a number greater than 4' or 'a tail and a number not exceeding 3' is:

A $\frac{1}{24}$ **B** $\frac{1}{6}$ **C** $\frac{1}{4}$ **D** $\frac{5}{12}$

4 A red die and a blue die are rolled on a table. Find the probability of each event:

(a) the same number with each die
(b) the sum of the numbers exceeds 9
(c) a 3 with the red die and a 4 with the blue die
(d) an odd number with the red die and an even number with the blue die
(e) the sum of the numbers is less than 2.

5 A coin is tossed three times. A is the event 'at least two tails'; B is the event 'three heads or three tails'; C is the event 'at least one tail'. Which of the following are independent?

(a) A and B **(b)** A and C **(c)** B and C

6 A die is rolled and a number is selected at random from the set {1, 2, 3, 4, 5}. The die number and the selected number are added together to find the score. Find the probability of each event:

(a) A: an even number is selected from the set **(b)** B: the die rolls an odd number
(c) C: the score exceeds 9 **(d)** D: the score is 10

7 A die is loaded so that the probability of rolling a 6 is $\frac{3}{10}$, the probability of a 5 is $\frac{3}{10}$, and each of the other numbers is equally likely. If the die is rolled twice, find the probability of:

(a) two 6s **(b)** no 6s **(c)** at least one 6 **(d)** the sum of the two numbers being six.

8 In a large school, 25% of the students ride bicycles to school and 40% of the students have fair hair. One student is selected at random. What is the probability that the student:

(a) has fair hair and rides a bicycle to school
(b) does not have fair hair and does not ride a bicycle to school
(c) has fair hair but does not ride a bicycle to school
(d) rides a bicycle to school but does not have fair hair?

9 For a certain species of bird, there is a chance of 4 in 5 that a fledgling will survive the first month after birth. From a brood of 3 fledglings, what is the probability that:

(a) all will survive **(b)** none will survive **(c)** at least one will survive?

10 One card is drawn at random from a standard deck of 52 playing cards. The card is replaced in the deck and the deck is shuffled. A second card is then drawn. What is the probability that:

(a) both cards are diamonds **(b)** neither card is a diamond
(c) only one of the cards is a diamond **(d)** only the first card is a diamond
(e) only the second card is a diamond **(f)** at least one of the cards is a diamond?

11 Cube A has 4 red faces and 2 white faces; cube B has 3 red faces and 3 white faces; cube C has 2 red faces and 4 white faces. The three cubes are rolled like dice. What is the probability of rolling:

(a) 3 red faces **(b)** 3 white faces **(c)** red with A and B, white with C
(d) red with A, white with B and C **(e)** at least one red face?

12 An athlete competes in races over 100 m, 200 m and 400 m, and she estimates her chances of winning as $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Using these probabilities, calculate the probability that:

(a) she wins all three races **(b)** she loses all three races
(c) she wins the 100 m race and loses the others **(d)** she wins the 400 m race and loses the others
(e) she wins the 100 m race and the 200 m race but loses the 400 m race.

13 A container holds 2 blue, 3 black and 5 white balls. A ball is withdrawn and then replaced in the container. This is repeated three times. Find the probability of each event:

(a) 3 white balls **(b)** a black ball in the first two drawings, but not in the third
(c) white, black, blue (in that order) **(d)** white, black, white (in that order)
(e) not more than two white balls **(f)** a white or a black ball each time.

14 A coin is tossed four times. What is the probability of:

(a) 4 heads **(b)** 4 tails **(c)** head, tail, head, tail (in that order)
(d) heads in the first 3 tosses but not in the fourth **(e)** heads in any one of the 4 tosses?

15 A student estimates that the chances of passing English, Mathematics and Physics are respectively $\frac{9}{10}$, $\frac{4}{5}$ and $\frac{3}{4}$. Estimate the probability of the events:

(a) passes English only **(b)** passes English and Mathematics only
(c) passes all three subjects **(d)** fails every subject.

16 A survey finds that, on average, 2 out of every 3 people interviewed are in favour of a certain proposal. If a random group of 3 people are interviewed, what is the probability that:

(a) all will be in favour of the proposal **(b)** none will be in favour
(c) the first and third people interviewed will be in favour, but not the second?

17 One container holds 2 red cubes and 4 blue cubes and a second container holds 4 red cubes and 3 blue cubes. One cube is selected at random from each of the two containers. What is the probability that one of the cubes is red?

18 The probability that a certain woman will be alive in 20 years is $\frac{2}{3}$ and the probability that a certain man will be alive is $\frac{3}{5}$. What is the probability that in 20 years' time:

(a) both will be alive (b) only one will be alive (c) at least one will be alive?

19 There are three containers A, B and C. A contains 3 black and 2 white cubes; B contains 3 black and 1 white cube; C contains 3 black and 3 white cubes.

(a) A container is chosen at random and from it a cube is chosen at random. Illustrate this two-stage process with a probability tree diagram.

(b) What is the probability that the cube is black?

20 On average, a student misses her bus to school once every eight weeks (where there are five days per school week). Find the probability that she:

(a) does not miss the bus on any one morning (b) catches the bus on two successive mornings

(c) catches the bus each morning for a week (5 days) (d) misses the bus on at least one morning in a week.

21 To open a locked safe requires a correct 3-digit combination. If a combination is chosen at random, calculate the probability of:

(a) succeeding at the first attempt (b) failing at the first attempt.

22 The first race at Randwick has 13 horses running and the second race has 16 horses. Assuming that all horses have an equal chance of winning, calculate the probability of predicting:

(a) a double (i.e. a winner in each race)

(b) a quinella (i.e. the two horses that come first and second, in either order) in the first race

(c) a quinella in the second race

(d) a quinella in both races.

23 To gain a driver licence in NSW, you must pass both a touch-screen computer-based test and a practical driving test. Statistics show that 70% of learners pass the computer-based test on the first attempt; of those who fail, 90% pass on the second attempt. Also, 60% of learners pass their first practical test and 80% pass their second practical test. The computer-based and practical tests are independent. Calculate the probability of:

(a) passing the computer-based test on the second attempt

(b) passing the computer-based test after no more than two attempts

(c) needing to take a third computer-based test

(d) passing the practical test on the second attempt

(e) receiving a licence after taking one practical test and two computer-based tests.

9.6 DEPENDENT EVENTS

Two events are **dependent** if the outcome of one event is affected by the outcome of the other event. This means that the probability of both events together is *not* the same as the probability of both events separately. Their probabilities cannot be simply multiplied together as if they are separate successive outcomes (as for independent events).

In other words:

If events A and B are dependent, then: $P(A \text{ and } B) \neq P(A) \times P(B)$

Example 17

Two dice are rolled. A is the event '5 with the first die' and B is the event 'sum of the numbers on the two dice exceeds 10'.

(a) Find $P(A \text{ and } B)$ (also written $P(A \cap B)$).

(b) Are A and B independent?

(c) Find the probability that the sum of the numbers exceeds 10, given that the first die rolls a 5.

Solution

(a) Rolling two dice has 36 possible outcomes.

$A = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$, so $P(A) = \frac{6}{36} = \frac{1}{6}$

$B = \{(5, 6), (6, 5), (6, 6)\}$, so $P(B) = \frac{3}{36} = \frac{1}{12}$

A and $B = A \cap B = \{5, 6)\}$, so $P(A \text{ and } B) = \frac{1}{36}$

Note that in this case $P(A \text{ and } B) \neq P(A) \times P(B)$.

(b) Because $P(A \text{ and } B) \neq P(A) \times P(B)$, events A and B are not independent. You say they are **dependent**.

(c) You are asked to find the probability of event B given that event A has happened. This can be written as $P(B \mid A)$.

Because A has occurred, the only possible events in this case are (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6). Of these 6 outcomes, only one is favourable: (5, 6).

Hence $P(B \mid A) = \frac{1}{6}$.

For dependent events:

$$P(A \cap B) = P(A) \times P(B \mid A)$$

where $P(B \mid A)$ is the probability of B occurring given that A has already occurred.

This statement can also be written as: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, $P(A) \neq 0$

Sampling without replacement from a small population

Example 18

From a set of 5 cards numbered 1, 2, 3, 4, 5, two cards are selected at random **without replacement** (i.e. without putting the first card back before taking the second). What is the probability that both cards are odd-numbered cards?

Solution

Let A be the event 'odd number first card' and B the event 'odd number second card'. You must find $P(A \text{ and } B)$. First: $P(A) = \frac{3}{5}$

If the first card is odd, then this leaves 2 odd cards in the remaining 4 cards, hence: $P(B \mid A) = \frac{2}{4}$

Thus: $P(A \text{ and } B) = P(A) \times P(B \mid A) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

It may have been easier to say $P(\text{odd, odd}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$.

$P(B \mid A)$ is called conditional probability.

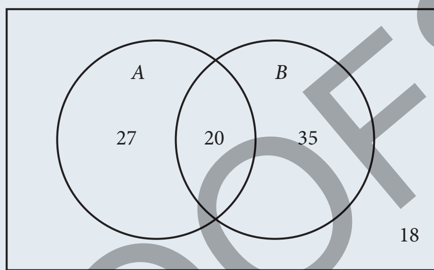
$$P\left(A \mid B\right)=\frac{P(A \cap B)}{P(B)},\ |B| \neq 0$$

Thinking back to the set notation used earlier, $P\left(A \mid B\right)=\frac{n(A \cap B)}{n(B)}, n(B) \neq 0$, and $P\left(B \mid A\right)=\frac{n(A \cap B)}{n(A)}, n(A) \neq 0$.

Example 19

The same information is given in two different formats, a table and a Venn diagram. Using each form of information, find (a) $P(A \mid B)$ (b) $P(B \mid A)$.

	A	$\tilde{A}$	Total
B	20	35	55
$\tilde{B}$	27	18	45
Total	47	53	100



Solution

(a) From the table, $n(B) = 55$ and $n(A|B) = 20$, the number in B that are also in A.

Hence $P\left(A \mid B\right)=\frac{n\left(A \mid B\right)}{n\left(B\right)}=\frac{20}{55}=\frac{4}{11}$

From the Venn diagram, $n(A) = 47$ and $n(A \cap B) = 20$, the number in both A and B.

Hence $P\left(A \mid B\right)=\frac{n\left(A \cap B\right)}{n\left(B\right)}=\frac{20}{55}=\frac{4}{11}$

(b) From the table, $n(A) = 47$ and $n(B \mid A) = 20$, the number in A that are also in B.

Hence $P\left(A \mid B\right)=\frac{n\left(A \mid B\right)}{n\left(B\right)}=\frac{20}{47}$

From the Venn diagram, $n(A) = 47$ and $n(B \cap A) = 20$, the number in both A and B.

Hence $P\left(A \mid B\right)=\frac{n\left(A \cap B\right)}{n\left(B\right)}=\frac{20}{47}$

Note: If $n(A \cap B) = 0$ then the events A and B are independent.

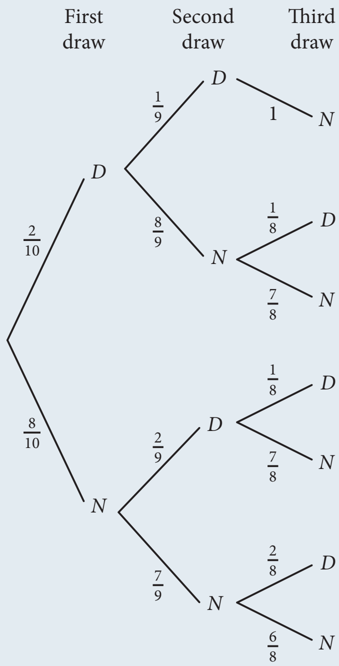
Example 20

A carton contains 10 batteries, 2 of which are defective. A sample of 3 batteries is drawn at random from the carton. Find the probability that not more than one battery is defective, if the sampling is done:

(a) without replacement **(b)** with replacement.

Solution

(a) Let D denote 'defective' and N 'non-defective'. You can represent the outcomes using a tree diagram:



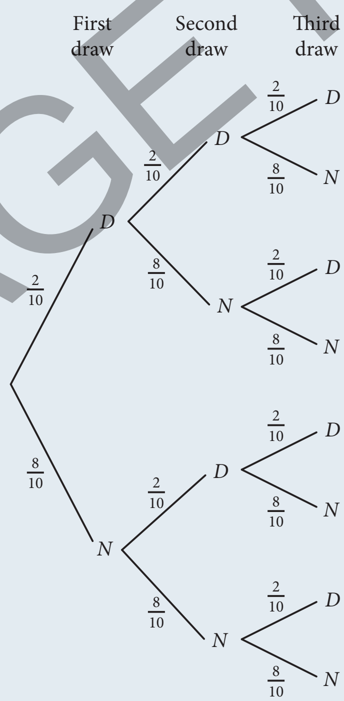
Possible Outcomes	Probability
DDN	$\frac{2}{10} \times \frac{1}{9} \times 1 = \frac{1}{45}$
DND	$\frac{2}{10} \times \frac{8}{9} \times \frac{1}{8} = \frac{1}{45}$
DNN	$\frac{2}{10} \times \frac{8}{9} \times \frac{7}{8} = \frac{7}{45}$
NDD	$\frac{8}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{45}$
NDN	$\frac{8}{10} \times \frac{2}{9} \times \frac{7}{8} = \frac{7}{45}$
NND	$\frac{8}{10} \times \frac{7}{9} \times \frac{2}{8} = \frac{7}{45}$
NNN	$\frac{8}{10} \times \frac{7}{9} \times \frac{6}{8} = \frac{21}{45}$

There is no branch for DDD because there are only 2 defective batteries in the carton.

The statement 'not more than 1 defective' is the same as '0 defective or 1 defective'.

$$\therefore \text{Required probability} = P(DNN) + P(NDN) + P(NND) + P(NNN)$$
$$= \frac{7}{45} + \frac{7}{45} + \frac{7}{45} + \frac{21}{45} = \frac{42}{45} = \frac{14}{15}$$

(b) Because each battery is replaced in the carton before the next withdrawal is made, the probability of a defective battery being withdrawn remains the same each time: $\frac{2}{10}$ for each withdrawal. You can again represent the outcomes using a tree diagram (note the difference compared to part **(a)**):



Possible Outcomes	Probability
DDD	$\frac{2}{10} \times \frac{2}{10} \times \frac{2}{10} = \frac{8}{1000}$
DDN	$\frac{2}{10} \times \frac{2}{10} \times \frac{8}{10} = \frac{32}{1000}$
DND	$\frac{2}{10} \times \frac{8}{10} \times \frac{2}{10} = \frac{32}{1000}$
DNN	$\frac{2}{10} \times \frac{8}{10} \times \frac{8}{10} = \frac{128}{1000}$
NDD	$\frac{8}{10} \times \frac{2}{10} \times \frac{2}{10} = \frac{32}{1000}$
NDN	$\frac{8}{10} \times \frac{2}{10} \times \frac{8}{10} = \frac{128}{1000}$
NND	$\frac{8}{10} \times \frac{8}{10} \times \frac{2}{10} = \frac{128}{1000}$
NNN	$\frac{8}{10} \times \frac{8}{10} \times \frac{8}{10} = \frac{512}{1000}$

$$\therefore \text{Required probability} = P(DNN) + P(NDN) + P(NND) + P(NNN)$$
$$= \frac{128}{1000} + \frac{128}{1000} + \frac{128}{1000} + \frac{512}{1000} = 0.896$$

Alternatively:

$$\begin{aligned}\text{Required probability} &= 1 - P(2 \text{ defectives} + 3 \text{ defectives})\\ &= 1 - [P(DDN) + P(DND) + P(NDD) + P(DDD)]\\ &= 1 - \left[\frac{32}{1000} + \frac{32}{1000} + \frac{32}{1000} + \frac{8}{1000}\right]\\ &= 0.896\end{aligned}$$

Sampling without replacement from a large population

Example 21

A shoe manufacturer makes two types of shoes: sneakers and boots. Of the shoes at the factory, 60% are sneakers. If a random sample of 3 pairs of shoes is taken from the factory, find the probability that the sample is such that:

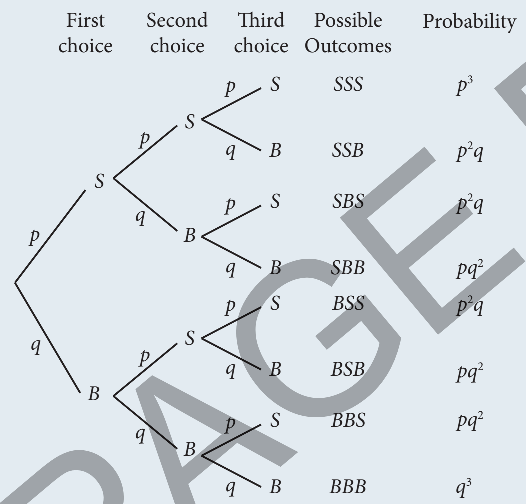
(a) there are exactly 2 pairs of sneakers

(b) there is at least 1 pair of boots

(c) the first two pairs chosen are sneakers and the third pair are boots.

Solution

Let $p = 0.6$ and $q = 0.4$ be the respective probabilities of choosing a pair of sneakers and a pair of boots.



(a) $\text{Required probability} = P(SSB) + P(SBS) + P(BSS)$
$= 3p^2q$
$= 3 \times 0.6^2 \times 0.4$
$= 0.432$

(b) $\text{Required probability} = 1 - P(0 \text{ boots})$
$= 1 - P(SSS)$
$= 1 - p^3$
$= 1 - 0.6^3$
$= 0.784$

(c) $\text{Required probability} = P(SSB)$
$= p^2q$
$= 0.6^2 \times 0.4$
$= 0.144$

EXPLORE FURTHER

Sampling without replacement

Use a spreadsheet to explore sampling without replacement from both small and large populations.

EXERCISE 9.6 DEPENDENT EVENTS

1 A box contains 5 black cubes and 3 red cubes. Two cubes are drawn at random from the box. Find the probability that:

(a) both cubes are black (b) both cubes are the same colour

(c) both cubes are different colours.

2 A box contains 6 green balls and 4 white balls. A batch of two balls is drawn at random from the box. The probability that the two balls are the same colour is:

A $\frac{1}{3}$ **B** $\frac{7}{15}$ **C** $\frac{2}{15}$ **D** $\frac{2}{45}$

3 An angler has caught 15 fish, of which 3 are undersized. A random sample of 3 fish is taken without replacement by an inspector. The angler is fined if one or more of the fish in the sample is undersized. What is the probability that the angler is fined?

4 A carton contains a dozen eggs, 3 of which have a double yolk. If 3 eggs are taken to make a cake, find the probability that all 3 eggs will have double yolks.

5 In a group of 9 people, 3 have brown hair and 6 have black hair. A random sample of 2 people is selected from the group. What is the probability that:

(a) the first person selected has black hair **(b)** both people selected have black hair
(c) one person has brown hair and the other has black hair?

6 From a standard deck of 52 playing cards, two cards are selected at random without replacement. Indicate whether each statement below is correct or incorrect.

(a) $P(\text{both cards are diamonds}) = \frac{1}{17}$ **(b)** $P(\text{both cards are the same suit}) = \frac{1}{17}$
(c) $P(\text{one card is a spade and the other is a club}) = \frac{13}{102}$ **(d)** $P(\text{both cards are different suits}) = \frac{13}{17}$

7 A container holds 3 white balls, 4 red balls and 5 black balls. Two balls are drawn at random without replacement. What is the probability that the balls are:

(a) both white **(b)** both red **(c)** both black?

8 From 7 teachers and 5 pupils, a random selection of 2 people is made. What is the probability that:

(a) they are both teachers **(b)** they are both pupils **(c)** one is a teacher and the other is a pupil?
(d) How can the answer to **(c)** be deduced from the answers to **(a)** and **(b)**?

9 From the letters of the word 'PROMISE', three letters are chosen at random, one at a time. What is the probability that the three letters are:

(a) vowels **(b)** consonants **(c)** vowel, consonant, vowel (in that order)?

10 A punter correctly picked first and second in a race of 10 horses. What is the probability of this, if all the horses were equally likely to win?

11 In a raffle, 20 tickets are sold and there are 2 prizes. If you buy 5 tickets, what is the probability that you win at least one of the prizes?

12 Group *A* contains 10 gorillas and 5 chimpanzees. Group *B* contains 4 gorillas and 6 chimpanzees. Two apes are selected at random from the groups. What is the probability that:

(a) they are both gorillas, if they are selected from Group *A*
(b) they are both chimpanzees, if they are selected from Group *B*
(c) one is a gorilla, the other is a chimpanzee, if one is selected from each group?

13 A box of 10 chocolates contains 4 hard-centred and 6 soft-centred chocolates. If two chocolates are selected at random, what is the probability that:

(a) they both have hard centres **(b)** they both have soft centres
(c) one has a soft centre and the other has a hard centre?

14 In a lottery game, three numbers are selected at random from 1, 2, 3, 4, … 40. Find the probability that the three selected numbers are all even.

15 A sample of three items is selected at random without replacement from a batch of ten items, four of which are defective. Find the probability that there is at most one defective item in the sample.

16 In a race with nine runners, if every possible order of finishing is equally likely, find the probability of picking the runners who come first, second and third in the correct order.

17 Three cards are drawn at random from a standard deck of 52 playing cards. What is the probability that 3 aces are drawn, if the drawing is done:

(a) without replacement (b) with replacement?

18 A bag contains a large number of five-cent coins and ten-cent coins in a ratio of 2 to 3. If three coins are randomly selected from the bag, find the probability that:

(a) two of them are five-cent coins (b) at least two of them are five-cent coins
(c) not more than two of them are five-cent coins.

19 A manufacturer finds that 10% of the products made in a factory are defective. If three products are taken at random, what is the probability that:

(a) all are defective (b) none are defective (c) more are defective than are non-defective?

20 A hand of three cards dealt from a standard deck of 52 playing cards contains the ace of clubs. What is the probability of this happening?

21 In a raffle, 30 tickets are sold and there are two prizes. What is the probability that a person who buys five tickets wins:

(a) neither prize (b) both prizes (c) at least one prize?

22 From a set of 10 cards numbered 1 to 10, two cards are drawn at random without replacement. What is the probability that:

(a) both numbers are even (b) one is even and the other is odd
(c) the sum of the two numbers is 12?

23 It is known that 7 out of 10 students from a certain school will go on to university. If a group of 3 students is chosen at random from this school, find the probability that:

(a) all will go on to university (b) some will go on to university.

24 An archer finds that her ratio of success to failure in hitting a bullseye is 9 to 1. If 3 arrows are shot, what is the probability of:

(a) 3 successes (b) at least 2 successes (c) not more than 1 success?

25 On average, at a particular beach it rains on 2 days out of every 7. Find the probability that on a given weekend it will rain:

(a) on both days (b) on at least one day.

26 A container holds a number of cubes: 60% are white and the remainder are black. Two cubes are randomly selected without replacement. What is the probability that:

(a) they are the same colour (b) they are different colours?

27 In a large flock of birds, 50% are red and 50% are green. If two birds are selected at random, what is the probability that:

(a) they are both red (b) at least one is green?

28 Container X holds 1 white cube and 2 black cubes. Container Y holds 2 white cubes and 1 black cube. A container is selected at random and from it two cubes are selected without replacement. Draw a tree diagram to represent this three-stage process and find the probability that both cubes drawn are:

(a) the same colour (b) different colours.

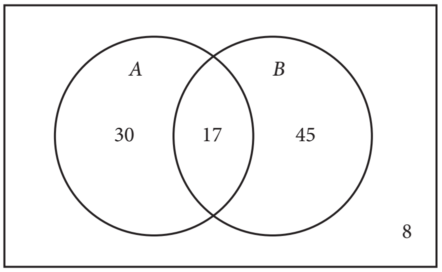
29 A and B play a 'set' of tennis: when a player wins two games, the set is won. If A has probability 0.6 of winning any one game, what is the probability that A wins the set? Construct a tree diagram.

30 Of two coins A and B, A is a fair coin while B is loaded with a probability of 0.6 for heads. A coin is chosen at random and tossed twice. What is the probability of tossing two heads?

31 Given that $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$, find:

(a) $P(A \mid B)$ (b) $P(B \mid A)$.

32 Using the information given in the diagram, find (a) $P(A|B)$ (b) $P(B|A)$.



33 (a) A coin is tossed 4 times and it is recorded that it lands heads exactly 3 times. What is the probability that the first toss was a tail?

(b) A pair of fair dice are rolled. Find the probability that both the numbers shown are even given that the first die shows a number more than 4.

(c) You select a number from the set of numbers, 1 to 100. What is the probability that the number selected is divisible by 4 if you know that it is divisible by 6?

34 Find (a) $P(A \cap B)$ and (b) $P(B|A)$ if:

(a) $P(A) = 0{,}8$, $P(B) = 0.3$ and $P(A \cup B) = 0.75$

(b) $P(A) = 0.35$, $P(B) = 0.55$ and $P(A \cup B) = 0.8$

(c) $\mathrm{P(A)} = 0.45$, $\mathrm{P(B)} = 0.5$ and $\mathrm{P(A \cup B)} = 0.9$

35 The faces of a die have a colour on them. 3 faces are red, 2 faces are blue and one face is green. The die is rolled twice. The probability of obtaining one red if we know that the first roll is not red is:

A $\frac{2}{5}$ B $\frac{3}{5}$ C $\frac{1}{2}$ D $\frac{1}{5}$

9.7 DISCRETE RANDOM VARIABLE

When performing a sampling procedure, a number of different outcomes are expected. For example, when rolling a normal die a large number of times, you can expect to observe some of each of the values from {1, 2, 3, 4, 5, 6}. The outcome can vary between rolls. X, the observed outcome, is a **random variable**. In particular, X is a **discrete random variable** because the list of possible outcomes is countable. Discrete random variables are often associated with number or size.

On the other hand, if the list of possible outcomes is not countable, the variable is a **continuous random variable**. For example, when measuring the heights of a sample of people, although you might expect the measurements to fall within a range (say, 140 cm to 190 cm), each individual value is dependent only on the degree of accuracy of the measuring instrument.

Continuous random variables are often associated with height, mass and time. Further work will be done with continuous random variables in a later chapter.

Discrete variables and whole numbers

A discrete random variable is not restricted to taking on whole number values; the important criterion is that the number of outcomes must be countable. For example, shoe sizes using the British measuring system increase in half sizes such as 7, $7\frac{1}{2}$, 8, . . ., but since the total number of different sizes can be counted, the variable is discrete.

The number of customers who purchase coffee at a café between 7 am and 9 am is a discrete random variable.

The time it takes for a battery to charge is a continuous random variable.

Displaying data

In this section you will look at adding information to a Frequency Table by adding columns for relative frequency, cumulative frequency and cumulative relative frequency.

This information will then be graphed as relative frequency and cumulative frequency histograms and polygons.

The above will then be used to find the mode and the median and you will then use the relative frequency to estimate the probability of results in experiments.

Frequency tables

A **frequency table** is used to summarise data. It can show frequency values for individual data values or grouped data. When grouped, the data may represent discrete data or continuous data.

Individual data

Number of TVs	Frequency
0	1
1	4
2	9
3	7
4	3

Grouped discrete data

Number of CDs	Frequency
0–9	3
10–19	5
20–29	6
30–39	2
40–49	1

Grouped continuous data

Distance travelled (km)	Frequency
0–<1	3
1–<2	5
2–<3	6
3–<4	4
4–<5	1

For the continuous data example, 1–<2 means responses from 1 km to less than 2 km. So, this would include distances such as 1.2 km, 1.23 km, 1.235 km and so on to 1.999… km. A distance of 2.0 km is recorded in the next group.

Relative frequency

The elements in the relative frequency column of a frequency table are obtained by dividing each frequency by the total frequency.

The relative frequency of an event gives the experimental probability of that event occurring. The sum of the relative frequencies is 1.

The cumulative relative frequency column has also been added. This gives the relative frequency less than or equal to a score or less than the upper score in grouped continuous data.

The earlier table now becomes:

Individual data

Number of TVs	Frequency	Relative frequency	Cumulative relative frequency
0	1	$\frac{1}{24}=0.0417$	$\frac{1}{24}=0.0417$
1	4	$\frac{4}{24}=0.1667$	$\frac{5}{24}=0.2083$
2	9	$\frac{9}{24}=0.375$	$\frac{14}{24}=0.5833$
3	7	$\frac{7}{24}=0.2917$	$\frac{21}{24}=0.875$
4	3	$\frac{3}{24}=0.125$	$\frac{24}{24}=1$
Total	24	1	

The relative frequency of a score of 3 or less is 0.875.

Grouped data

Number of CDs	Frequency	Relative frequency	Cumulative relative frequency
0–9	3	$\frac{3}{17}=0.1765$	$\frac{3}{17}=0.1765$
10–19	5	$\frac{5}{17}=0.2941$	$\frac{8}{17}=0.4706$
20–29	6	$\frac{6}{17}=0.3529$	$\frac{14}{17}=0.8235$
30–39	2	$\frac{2}{17}=0.1176$	$\frac{16}{17}=0.9412$
40–49	1	$\frac{1}{17}=0.0588$	$\frac{17}{17}=1$
Total	17	1	

The relative frequency of a score of 29 or less is 0.8235.

Grouped continuous data

Distance travelled (km)	Frequency	Relative frequency	Cumulative relative frequency
0–<1	3	$\frac{3}{19} = 0.1579$	$\frac{3}{19} = 0.1579$
1–<2	5	$\frac{5}{19} = 0.2632$	$\frac{8}{19} = 0.4211$
2–<3	6	$\frac{6}{19} = 0.3158$	$\frac{14}{19} = 0.7368$
3–<4	4	$\frac{4}{19} = 0.2105$	$\frac{18}{19} = 0.9474$
4–<5	1	$\frac{1}{19} = 0.0526$	$\frac{19}{19} = 1$
Total	19	1	

The relative frequency of a score less than 3 is 0.7368.

Cumulative frequency

The cumulative frequency column gives the frequency less than or equal to a score or less than the upper score in grouped continuous data.

Individual data

Number of TVs	Frequency	Cumulative frequency
0	1	1
1	4	5
2	9	14
3	7	21
4	3	24
Total	24	

Grouped data

Number of CDs	Frequency	Cumulative frequency
0–9	3	3
10–19	5	8
20–29	6	14
30–39	2	16
40–49	1	17
Total	17	

Grouped continuous data

Distance travelled (km)	Frequency	Cumulative frequency
0–<1	3	3
1–<2	5	8
2–<3	6	14
3–<4	4	18
4–<5	1	19
Total	19	

EXPLORE FURTHER

Frequency tables

Explore how to create frequency tables using a spreadsheet to display frequency, relative frequency and percentage frequency.

Measures of central tendency—median

The median is the middle data value of an ordered data list. The method for finding the median depends on whether there is an odd or even number of data values.

For odd n, the median is the $\left(\frac{n+1}{2}\right)$th value.

For even n, the median is the average of the $\left(\frac{n}{2}\right)$th and the $\left(\frac{n+2}{2}\right)$th values.

When data is grouped, you should state the class interval in which the median falls.

Example 22

Find the median of the following data set.

(a) Number of cars that passed the school gate in a number of 5 minute periods.
10, 5, 12, 13, 14, 7, 12, 15, 7, 3, 2, 8

(b) Number of brothers and sisters for the students in two Year 11 classes.

Number of siblings	Frequency
0	7
1	11
2	15
3	3
4	1

Solution

(a) Rewrite the data in numerical order from smallest to largest:
2, 3, 5, 7, 7, 8, 10, 12, 12, 13, 14, 15

There is an even number of data values, so identify the $\left(\frac{n}{2}\right)$th and $\left(\frac{n+2}{2}\right)$th values:

$\left(\frac{n}{2}\right)$th $= \frac{12}{2} = 6$th value so count from the left until you get to the 6th value which is 8.

$\left(\frac{n+2}{2}\right)$th $= 7$th value which is 10.

The median is the average of these two values: $\frac{8+10}{2} = 9$
The median is 9.

(b) The frequency table has effectively put the data in order already. Find $\sum f$ to decide if you are dealing with an odd or even number of data values.

Number of siblings	Frequency
0	7
1	11
2	15
3	3
4	1
	$\sum f = 37$

There is an odd number of data values so the median will be the $\left(\frac{n+1}{2}\right)$th value: $\left(\frac{n+1}{2}\right)$th value $= \frac{37+1}{2} = 19$

Count down the frequency column to find the required value and state the answer: 19th value is 2.

The median number of brothers and sisters is 2.

Example 23

Find the median of the following data set, which shows the time taken by the competitors in the under-17 100 m dash at the school athletics carnival.

Time (x)	Frequency (f)
12–<12.5	3
12.5–<13	5
13–<13.5	12
13.5–<14	28
14–<14.5	10

Solution

The frequency table has ordered data. Find $\sum f$ and see whether there is an odd or even number of data values:

Time (x)	Frequency (f)
12–<12.5	3
12.5–<13	5
13–<13.5	12
13.5–<14	28
14–<14.5	10
	$\sum f = 58$

There is an even number of data values, so the median will be the average of the $\left(\frac{n}{2}\right)$th and $\left(\frac{n+2}{2}\right)$th values:

$\left(\frac{n}{2}\right)$th value $= \frac{58}{2} = 29$th value.

Count down the frequency column until you get to the 29th value: in the interval 13.5–<14

$\left(\frac{n+2}{2}\right)$th value $= 30$th value

Count down the frequency column until you get to the 30th value: in the interval 13.5–<14

The median is in the interval 13.5–<14.

Measures of central tendency—mode

The mode is the most frequently occurring data value. A data set with a unique mode is sometimes referred to as **unimodal**. If a data set has two results with the same equal highest frequency, then the data is **bimodal**. If more than two results share the highest frequency we usually say the data has no mode. Sometimes this is described as **multimodal**.

Example 24

Find the mode of the following data sets.

(a) The number of cars that passed the school gate in a number of 5 minute periods:
10, 5, 12, 13, 14, 7, 12, 15, 12, 3, 2, 8

(b) The number of brothers and sisters for the students in two Year 11 classes:

Number of siblings (x)	Frequency (f)
0	7
1	11
2	15
3	3
4	1

Solution

(a) Write the data in order to make it easier to identify the mode: 2, 3, 5, 7, 8, 10, 12, 12, 12, 13, 14, 15
The score of 12 has the highest frequency 3 so the mode is 12.

(b) Identify the highest frequency from the frequency table: The score 2 has a frequency of 15.
The mode is 2.

Example 25

Find the mode of the following data set, which shows the time taken by the competitors in the under-17 100 metre dash at the school athletics carnival.

Time (x)	Frequency (f)
12–<12.5	3
12.5–<13	5
13–<13.5	12
13.5–<14	28
14–<14.5	10

Solution

Identify the highest frequency from the frequency table: The modal class is 13.5–<14

Technology does not generally display the mode as one of the summary statistics. However, it is easy enough to find without the use of technology.

EXERCISE 9.7 DISCRETE RANDOM VARIABLE

1 Information about the numbers from 1 to 50 inclusive is contained in the following table.

	Multiple of 3	Multiple of 7	Total
Odd	8	4	12
Even	8	3	11
Total	16	7	23

(a) How many multiples of 3 are there?
(b) Which two numbers have been included in two different groups?
(c) How many numbers from 1 to 50 inclusive are not included in this table?

2 A survey of 100 high school students was carried out asking what type of phone they owned. The results are contained in the following table.

	Apple	Samsung	LG	Other	Total
Boys	15	12	15	8	50
Girls	20	10	15	5	50
Total	35	22	30	13	100

(a) How many students had an Apple phone?
(b) How many students did not have a Samsung phone?
(c) What proportion of the students had a LG phone?
(d) If a student was selected at random from the group, what is the probability that their phone was not an Apple, Samsung or LG?

3 (a) Draw a histogram to represent the following set of continuous numerical data.

Score	Frequency
0–<10	9
10–<20	18
20–<30	13
30–<40	15
40–<50	14
50–<60	16

What are the lowest and highest scores that could have been included in the data?

(b) Draw a histogram to represent the following set of continuous numerical data.

Score	Frequency
141–<146	22
146–<151	31
151–<156	21
156–<161	26
161–<166	22
166–<171	24

What are the lowest and highest scores that could have been included in the data?

4 A survey of students asking them their favourite sport produced the information given in the following table.

	Football	Swimming	Basketball	Total
Male	32	30		70
Female		28	12	
Total	50		20	

(a) Copy the table and fill in the missing information.
(b) How many students gave swimming as their favourite sport?
(c) What proportion of the males gave basketball as their favourite sport?
(d) A student is chosen at random from the group. What is the probability that they gave football as their favourite sport?

5 The following data represents the number of kilometres travelled each day by a solar vehicle. The distances have been rounded to the nearest whole number of kilometres.

112, 95, 122, 78, 99, 145, 133, 160, 145, 144, 78, 66, 68, 79, 93, 125, 133, 89, 67, 78, 114, 134, 80, 65, 105, 115, 127, 134, 117, 84

(a) Given the way the data has been recorded would you consider it to be discrete or continuous? Give a brief justification for your answer.
(b) Construct a frequency table for the data, using a class interval of 10 and a starting value of 61.
(c) Looking only at the frequency table, what can you say about the maximum distance travelled on any one day?
(d) Construct a stem-and-leaf plot for the data.
(e) What can this stem-and-leaf plot tell us that the frequency table cannot?
(f) What does this question reinforce about stem-and-leaf plots compared to frequency tables?

6 A manufacturer received the following complaints from purchasers of their new ovens.

Defect	Frequency
Faulty light	17
Faulty fan	12
Faulty element	9
Damaged box	4
Scratched glass	3
Other	2
Total	**47**

(a) Add the percentage and cumulative percentage columns to this table.
(b) Draw the Pareto chart for this information.

7 A hotel chain conducted a survey of their guests and received the following responses.

Type of complaint	Frequency
Poor housekeeping	45
Slow check-in/checkout	25
Poor breakfast	15
Not value for money	9
Poor WiFi	6
Television	4
Parking	2
Total	**106**

(a) Add the percentage and cumulative percentage columns to this table.
(b) Draw the Pareto chart for this information.
(c) How could the hotel chain improve their customers' satisfaction?

8 Find the mode and the median for each set of scores.

(a) 1, 4, 3, 2, 6, 4, 8, 9, 4, 3, 1

(b) 2, 3, 6, 2, 9, 6, 9, 6, 3, 2, 5, 2

(c)

Score (x)	Frequency (f)
0	3
1	6
2	7
3	4
4	3
5	1

9 Find the mode and the median for each set of scores. Where necessary, state your answers correct to 2 decimal places.

(a) The following data represents the number of cars that passed the school gate in a number of 5-minute periods.
7, 14, 12, 11, 13, 16, 17, 9, 7, 12, 13, 15

(b) The following data represents the number of brothers and sisters for the students in two Year 11 classes.

Number of brothers and sisters (x)	Frequency (f)
0	6
1	11
2	18
3	9
4	3

(c) The following data represents the time taken by the competitors in the under-17 100 m dash at the school athletics carnival.

Time (x)	Frequency (f)
12–<12.5	4
12.5–<13	9
13–<13.5	15
13.5–<14	22
14–<14.5	17

10 Find the mode of the following data sets.

(a) 4, 6, 4, 2, 7, 8, 3, 4, 9, 1, 1

(b) 3, 5, 2, 1, 7, 4, 9, 3, 5, 6, 8

(c) 1, 4, 2, 1, 5, 4, 3, 1, 4, 2, 2

11 Find the median class interval for the following data sets.

(a)

Time (x)	Frequency (f)
12–<12.5	3
12.5–<13	5
13–<13.5	4
13.5–<14	11
14–<14.5	6

(b)

Time (x)	Frequency (f)
12–<12.5	6
12.5–<13	8
13–<13.5	4
13.5–<14	5
14–<14.5	7

12 Look at the following data set.

x	1	2	3	4	5	6
f	5	8	3	2	4	5

The mode and median (in that order) are:

A 2, 3 **B** 3.26, 3 **C** 3.26, 4 **D** 3, 3.26

13 Find the median for each of the following discrete data sets. Where necessary, state your answer correct to 2 decimal places.

(a)

Score	Frequency (f)
4	15
5	23
6	14
7	23
8	17
9	19
10	20

(b)

Score	Frequency (f)
201	49
202	83
203	99
204	121
205	143
206	117

14 The following table gives the maximum temperature (°C) reached on a particular day in a number of cities across the world. Answer the following questions about this data set.

City	Temp (°C)	City	Temp (°C)	City	Temp (°C)
Amsterdam	3	Copenhagen	0	New York	13
Athens	18	Helsinki	–4	Oslo	–3
Barcelona	9	Istanbul	18	Prague	0
Belgrade	3	London	5	Singapore	33
Budapest	1	Los Angeles	14	Taipei	27
Cairo	34	Madrid	6	Toronto	10
Chicago	13	Moscow	1	Warsaw	–1

(a) Find the mode of the temperatures. Give your answer correct to 2 decimal places.
(b) Calculate the median temperature.
(c) Draw a grouped data frequency table using –5 to <0 as the first class interval.
(d) In which class interval is the median temperature?

15 Consider the frequency tables:

(a)

Score	Frequency
44	8
45	12
46	13
47	10
48	9
49	5

(b)

Score	Frequency
101	11
102	14
103	18
104	19
105	13
106	12

(i) Copy the table and add a relative frequency column
(ii) What is the median for the data?
(iii) What is the probability that the score will be the:
(α) smallest one in the table? (β) largest one in the table?

16 Consider the frequency tables:

(a)

Score	Frequency
10–14	6
15–19	7
20–24	9
25–19	10
30–34	4
35–39	1

(b)

Score	Frequency
80–<90	16
90–<100	15
100–<110	17
110–<120	19
120–<130	12
130–<140	11

(i) Copy the table and add a cumulative frequency column.
(ii) Draw the cumulative frequency polygon for the data and use it to estimate the median (class).
(iii) What is the probability that the score will be in the:
(α) smallest class interval (β) largest class interval?

CHAPTER REVIEW 9

1 The combination lock on a safe has three concentric circular discs, each showing the digits 0 to 9. Only one combination of digits will open the safe. What is the probability of opening the safe at your first attempt if you do not know the combination?

2 One student has a pencil whose cross-section is square and whose faces are coloured black, white, green and red. Another student has a pencil whose cross-section is hexagonal (six-sided) and whose faces are coloured black, white, green, red, yellow and orange. Both pencils are rolled on a flat surface and the colours appearing uppermost are noted. Find the probability of each event:

(a) both colours uppermost are the same
(b) both colours uppermost are different
(c) black is uppermost on at least one of the pencils
(d) neither black nor white appears uppermost.

3 Four names beginning with B and five names beginning with G are in a hat. If two names are randomly drawn without replacement, find the probability that:

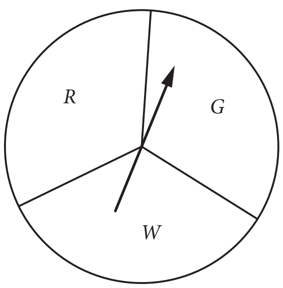
(a) two G names are drawn out
(b) at least one B name is drawn out.

4 A jar contains red buttons and white buttons in a ratio of 3 : 2. If three buttons are chosen at random from the jar, find the probability that:

(a) exactly two are red
(b) not more than one is white.

5 A box contains four balls marked 1, 2, 3, 4. A spinner has an equal chance of spinning *R*, *G* or *W*. A ball is selected at random from the box and the spinner is spun. Illustrate on a tree diagram the 12 possible outcomes.
Find the probability of each outcome:

(a) *R* and an odd number
(b) *W* and a number greater than 2, or *R* and an even number.



6 In a group of 25 students, 18 study Biology, 12 study Chemistry and 5 study neither Biology nor Chemistry. If a student is chosen at random, what is the probability that the student studies:

(a) Biology only
(b) Chemistry only
(c) Biology or Chemistry or both
(d) both Biology and Chemistry?

7 A coin is tossed and a die is rolled. What is the probability of 'a head and a number greater than 3' or 'a tail and a number not exceeding 4'?

8 Consider two spinners. One spinner shows the letters A, B, C, D and E, and the other spinner shows the digits 1, 2, 3, 4 and 5. When the spinners are spun, it is equally likely that they will stop on any letter or number. Find the probability of each event:

(a) B and an even number
(b) C or D, and an odd number
(c) E and an even number, or C and a number greater than 3
(d) a consonant and an odd number, or a consonant and a number greater than 2.

9 A die is rolled three times. What is the probability of:

(a) 3 sixes
(b) 0 sixes
(c) 3 odd numbers
(d) 3 even numbers
(e) a six in the first two tosses only
(f) a six, not a six, a six (in that order)?

10 Trinh and Oscar play three tennis matches. Trinh's chance of winning any one match is $\frac{2}{3}$. What is the probability that Trinh:

(a) wins all three matches
(b) loses all three matches
(c) wins the first and third but loses the second
(d) loses the first and wins the other two?

11 A man finds that he is late for work on 10% of occasions if he was on time the previous day, but late on 20% of occasions if he was late the previous day. Given that he was on time on Monday and worked on Tuesday, what is the probability that he is on time on Wednesday? Illustrate using a tree diagram.

12 A certain factory has three machines *A*, *B* and *C*, which manufacture 25%, 35% and 40% respectively of the factory's products. Of the machines' products, 5%, 4% and 2% respectively are defective. A product from the machines is selected at random. What is the probability that:

(a) it was manufactured by *A* and is defective **(b)** it was manufactured by *B* and is not defective?

13 From a group of 5 boys and 6 girls, two are selected at random for a class committee. What is the probability that a boy and a girl are selected?

14 A carton contains 10 electric lights, 3 of which are defective. Two are drawn at random. What is the probability that:

(a) the first drawn is defective **(b)** both are defective
(c) neither is defective **(d)** exactly one is defective?

15 Three cards are drawn at random from a standard deck of 52 playing cards. What is the probability that they are all from the same suit?

16 A bag contains 6 red balls and 4 white balls. A random sample of 3 is withdrawn. What is the probability that the balls are the same colour, if the sampling is done:

(a) without replacement **(b)** with replacement?

17 Three cards are taken without replacement from a standard deck of 52 playing cards. What is the probability of taking:

(a) exactly 3 hearts **(b)** exactly 3 aces **(c)** at least 1 heart?

18 A manufacturer of metal pistons finds that on average 20% of pistons are rejected because they are either oversize or undersize. What is the probability that a batch of three pistons will contain:

(a) no more than two rejects **(b)** at least two rejects?

19 In an assortment of bananas, the ratio of ripe bananas to unripe bananas is 3 to 5. If 3 bananas are chosen at random, find the probability that:

(a) exactly 2 will be ripe **(b)** at least 1 will be unripe.

20 In an opinion poll, the ratio of those in favour to those against a particular proposal was 7 to 3. If three randomly chosen people are interviewed about the proposal, what is the probability that:

(a) all will be in favour **(b)** the majority will be in favour
(c) not more than two will be against the proposal?

21 A container holds 10 cubes: 6 are white and the remainder are black. Two cubes are randomly selected without replacement. What is the probability that:

(a) they are the same colour **(b)** they are different colours?

22 In a group of 10 birds, 5 are red and 5 are green. If two birds are selected at random, what is the probability that:

(a) they are both red **(b)** at least one is green?

The data set shown is to be used for questions **23** and **24**.

x	1	2	3	4	5
f	3	5	2	4	2

23 Select the option closest to the median of the data set.

A 2.5 **B** 5 **C** 2.8 **D** 2

24 Select the option closest to the mode of the data set.

A 2.5 **B** 5 **C** 2.8 **D** 2

25 The height of every student in the school was recorded. The results are shown in the following table.

Height of student (cm)	Frequency
140–<145	5
145–<150	11
150–<155	18
155–<160	26
160–<165	44
165–<170	92
170–<175	64
175–<180	32
180–<185	17
185–<190	9
190–<195	4

(a) Copy the table and add a cumulative frequency column and a relative frequency column.
(b) What is the modal class?
(c) From the cumulative frequency column, estimate the median.
(d) What is the probability that a student's height falls in the range 175–<180 cm?