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MATHEMATICAL METHODS 12

UNITS 3 & 4

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STUDENT BOOK

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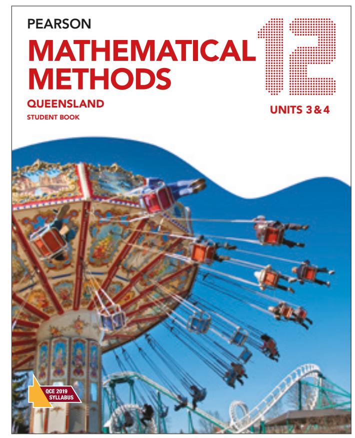

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Contents

Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non CAS)
- CASIO fx-CG50AU

Throughout the student book are Technology worked examples strategically placed within the theory for both the TI-Nspire CX (non CAS) and CASIO fx-CG50AU.

The examples clearly demonstrate how the technology can be used effectively and efficiently for the content being covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

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STUDENT BOOK

UNITS 3 & 4

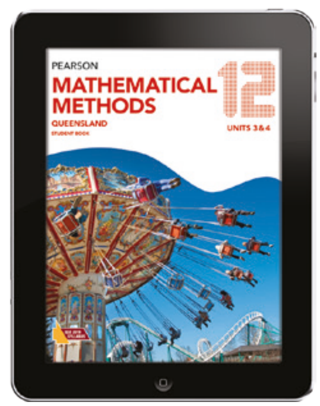
Mathematical Methods 12
Student book

Student book

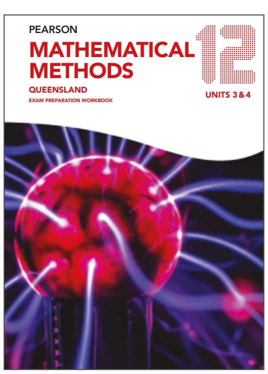
The student book has been written by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



Mathematical Methods 12
eBook



Mathematical Methods 12
Exam preparation workbook

Exam preparation workbook

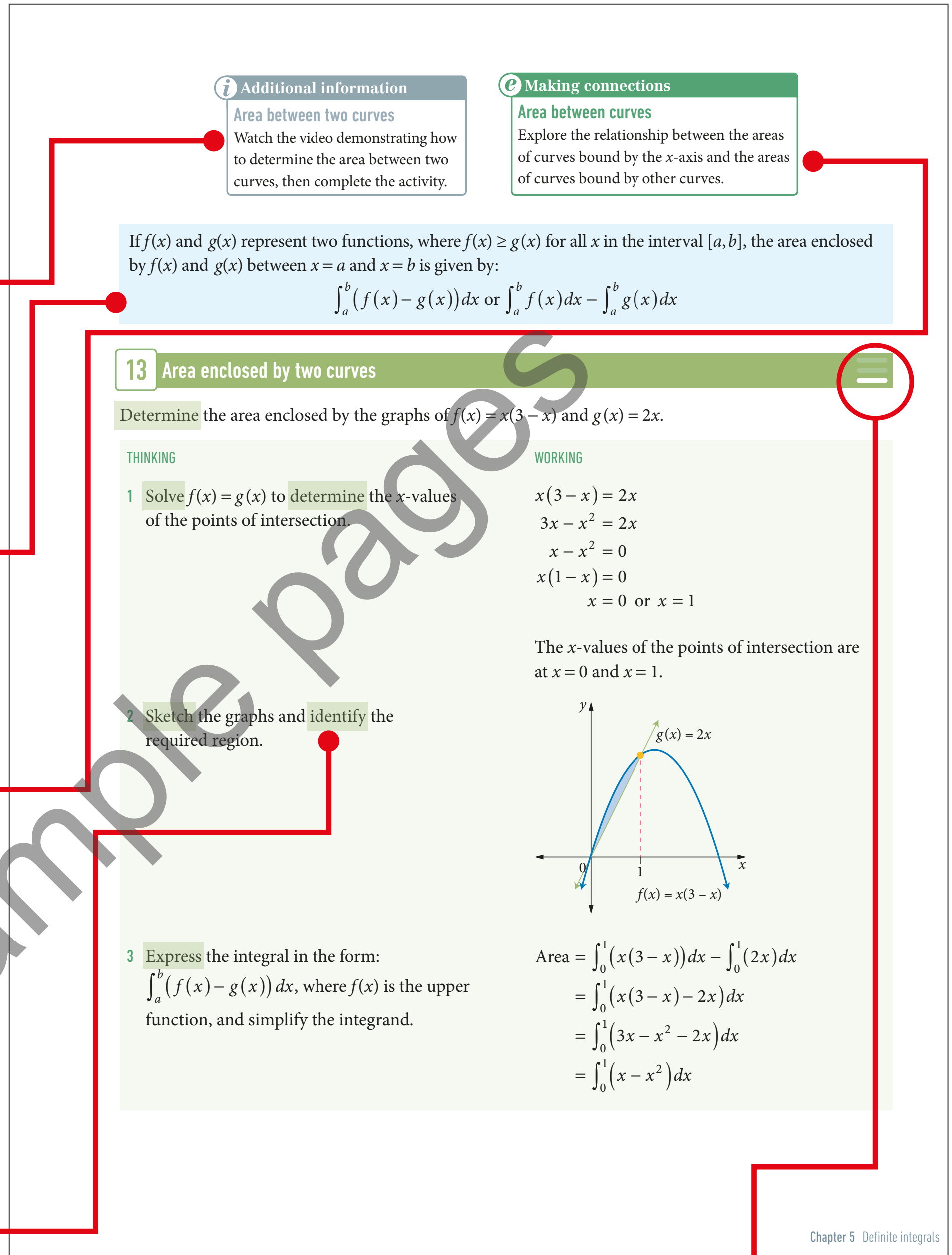
Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Mathematical Methods 12 Queensland Units 3 & 4

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teachers support.



Additional information

Area between two curves

Watch the video demonstrating how to determine the area between two curves, then complete the activity.

Making connections

Area between curves

Explore the relationship between the areas of curves bound by the x-axis and the areas of curves bound by other curves.

If $f(x)$ and $g(x)$ represent two functions, where $f(x) \geq g(x)$ for all x in the interval $[a, b]$, the area enclosed by $f(x)$ and $g(x)$ between $x = a$ and $x = b$ is given by:

$$\int_a^b \big(f(x) - g(x)\big)dx \text{ or } \int_a^b f(x)dx - \int_a^b g(x)dx$$

13 Area enclosed by two curves

Determine the area enclosed by the graphs of $f(x) = x(3 - x)$ and $g(x) = 2x$.

THINKING	WORKING
1 Solve $f(x) = g(x)$ to determine the x-values of the points of intersection.	$x(3-x) = 2x$ $3x - x^2 = 2x$ $x - x^2 = 0$ $x(1-x) = 0$ $x = 0$ or $x = 1$ The x-values of the points of intersection are at $x = 0$ and $x = 1$.
2 Sketch the graphs and identify the required region.	y, $g(x) = 2x$, 0, 1, x, $f(x) = x(3 - x)$
3 Express the integral in the form: $\int_a^b \big(f(x) - g(x)\big)dx$, where $f(x)$ is the upper function, and simplify the integrand.	Area $= \int_0^1 \big(x(3-x)\big)dx - \int_0^1 (2x)dx$ $= \int_0^1 \big(x(3-x) - 2x\big)dx$ $= \int_0^1 \big(3x - x^2 - 2x\big)dx$ $= \int_0^1 \big(x - x^2\big)dx$

Chapter 5 Definite integrals

Additional information

These interactives appear in the eBook in two forms, as videos explaining specific concepts or as interactive questions to check students' understanding.

Key information

Key information and rules are highlighted throughout the chapter.

Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Meeting the needs of the QCE Syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.

Making connections

The exponential function and its inverse

Explore the properties of the exponential graph and its inverse, the logarithmic graph.

Technology worked examples

These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Technology worked example

Solving exponential equations

Tech-free questions

These questions are designed to provide students with the opportunity to practise algebraic manipulations to prepare them for technology-free examination papers.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

EXERCISE 3.1 Logarithmic functions and their properties

1 Simplify each of the following expressions.

(a) $\log_4(1)$ (b) $2\log_4(1)$ (c) $\log_{10}(10)$ (d) $3\log_4(4)$

2 Which expression is equal to $\log_2(x) - \log_2(3x+2)$?

A $\log_2(4x+2)$ B $\log_2\left(3+\frac{3}{x}\right)$ C $\log_2(2x+2)$ D $\log_2\left(\frac{x}{3x+2}\right)$

3 Simplify each of the following expressions.

(a) $\log_5(3)+\log_5(5)$ (b) $\log_8(4)+\log_8(2)$ (c) $\log_3(5)-\log_3(4)$

(d) $\log_2(8)-\log_2(4)$ (e) $\log_{10}(6)-\log_{10}\left(\frac{3}{5}\right)$

(f) $\log_{10}(6)+\log_{10}(5)-\log_{10}(3)$ (g) $\log_2(12)-\log_2(2)-\log_2(3)$

4 Consider the expression $3\log_e(x)+2\log_e(x)+\log_e(4x)$.

(a) Which expression is equivalent to the original?

A $\log_e(2x^2+4x)$ B $\log_e(x^3+x^2+4x)$ C $\log_e(4x^6)$ D $6\log_e(4x)$

(b) Explain the common error made by a student who obtained an expression with a coefficient of 6 for part (a).

5 Simplify each of the following expressions.

(a) $\log_2(2^3)$ (b) $\log_3(27)$ (c) $\log_2\left(\frac{1}{2}\right)$

(d) $\log_{10}(0.01)$ (e) $3\log_8(2)$

(f) $2\log_e(\sqrt{e})$ (g) $-2\log_2\left(\frac{1}{4}\right)$

6 Convert each of the following to logarithmic form.

(a) $3^4=81$ (b) $10^3=1000$ (c) $2^{-2}=\frac{1}{4}$ (d) $10^{-3}=0.001$

7 Convert each of the following to exponential form.

(a) $\log_2(16)=4$ (b) $\log_3(27)=3$ (c) $\log_2\left(\frac{1}{8}\right)=-3$ (d) $\log_{10}(0.1)=-1$

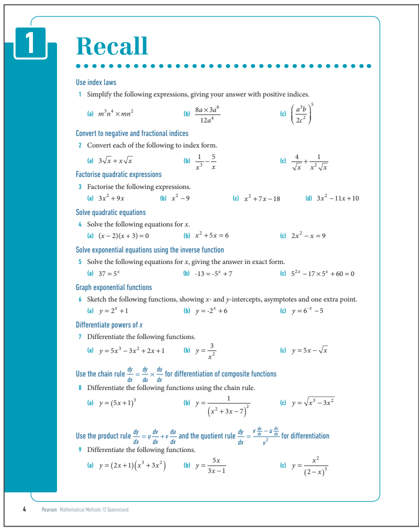
Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

WARNING

It is a common misconception that these expressions are equivalent.

$$\frac{\log_a(m)}{\log_a(n)} \neq \log_a(m)-\log_a(n) \qquad \log_a(m+n) \neq \log_a(m)+\log_a(n)$$



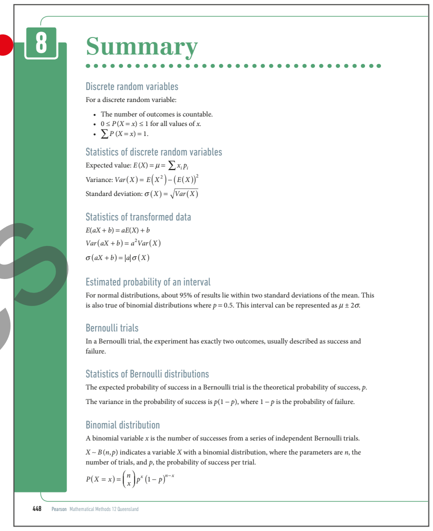
1 Recall

Recall

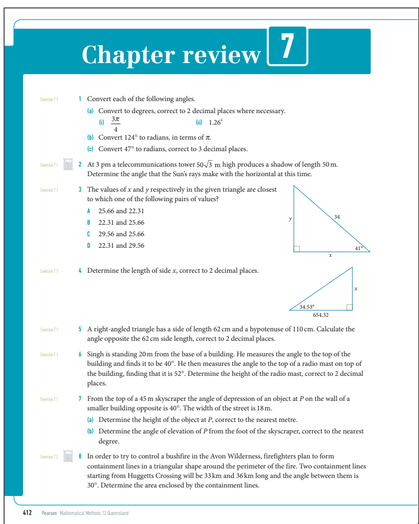
Each chapter begins with a review of assumed knowledge for the chapter.

Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.



8 Summary



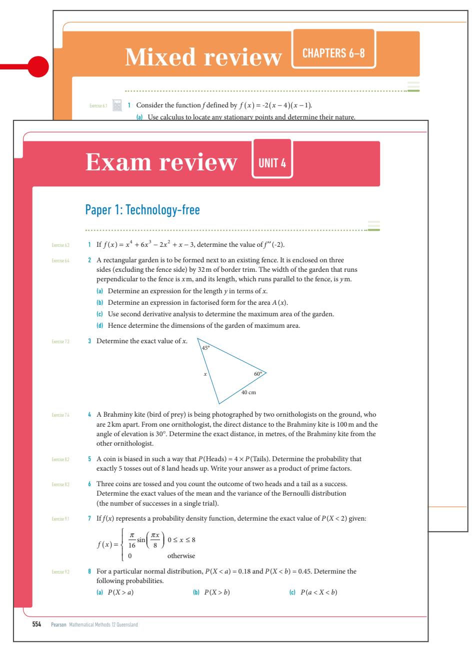
Chapter review 7

Chapter review

Every chapter review follows the QCAA examination proportions for level of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.

Mixed and exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each unit. Mixed reviews provide cumulative reviews placed midway through each unit.



Mixed review CHAPTERS 6–8

Exam review UNIT 4

Paper 1: Technology-free

4

Anti-differentiation

4 Recall

Differentiate functions of the form $f(x) = x^n$

1 Determine the derivative of each of the following.

(a) $y = 4x^3 - 2x^2 - 3x + 2$ (b) $y = (x+1)^2 + x^2$ (c) $f(x) = 3x^6 + 4x - 5$

(d) $f(x) = 2x^{\frac{2}{3}}$ (e) $y = \sqrt{x} - x^{-3} + \dfrac{4}{x}$

Determine a derivative at a given value

2 Determine the value of each of the following.

(a) $\dfrac{dy}{dx}$ if $y = 4 - 3x^3 + x^2$ and $x = -2$ (b) $f'(3)$ if $f(x) = 3x - 4x^2$

(c) $f'(16)$ if $f(x) = x^{-\frac{1}{2}} - 2x^{-\frac{1}{4}}$

Differentiate functions of the the form $f(x) = (ax + b)^n$

3 Determine the derivative of each of the following.

(a) $y = (3x+2)^4$ (b) $y = (2x+5)^3$ (c) $y = (2x-1)^{\frac{1}{2}}$

(d) $y = 5(3x-4)^{-2}$ (e) $y = (2x+3)^{\frac{3}{2}}$

Differentiate exponential and logarithmic functions

4 Differentiate the following functions.

(a) $y = e^{5x}$ (b) $y = e^{\frac{x}{2}}$ (c) $f(x) = 3e^{5x^2 - x}$

(d) $f(x) = 7\log_e(2x-1),\ x > \dfrac{1}{2}$ (e) $y = \log_e\left(\dfrac{x^2}{3+x}\right),\ x > -3$

Differentiate trigonometric functions

5 Determine the derivative of each of the following.

(a) $\sin(3x)$ (b) $\cos(5x)$ (c) $y = 3\sin\left(\dfrac{2x}{3}\right) + 2\cos(\pi x)$

(d) $f(x) = \sin(3 - 5x)$ (e) $f(x) = 3\cos(11x - 2)$

Differentiate using the product and quotient rule

6 Use the product or quotient rule to differentiate the following functions.

(a) $y = x^2\sin(x)$ (b) $y = x^4e^{2x}$ (c) $y = \dfrac{x^3}{\sin(x)}$

(d) $y = \dfrac{\log_e(x)}{e^{-x}}$ (e) $y = \dfrac{\sin(x)}{\cos(x)}$ (f) $y = \dfrac{\sqrt{x}}{x+5}$

Differentiate using the chain rule

7 Use the chain rule to determine the derivative of each of the following.

(a) $y = \left(3x^2 + 2x\right)^4$ (b) $y = \sin\left(x^2\right)$ (c) $y = \sqrt{x^3 - 4x + 2}$

(d) $y = e^{x^2}$ (e) $y = \sin^2(x)$

4.1 Introduction to anti-differentiation

An anti-derivative is a function $F(x)$, such that $F'(x) = f(x)$.
Anti-differentiating can be considered the reverse process of differentiating.
Hence anti-differentiation is the process of determining a function $f(x)$ from its derivative or gradient function $f'(x)$. The derivative of $3x^2$ is $6x$, so an anti-derivative of $6x$ is $3x^2$.

Differentiation $\longrightarrow$

$y = f(x)$	$\frac{dy}{dx} = f'(x)$

$\longleftarrow$ Anti-differentiation

Consider the following functions and their derivatives:

$f(x)$	$f'(x)$
$3x^2$	$6x$
$3x^2 + 2$	$6x$
$3x^2 - 7$	$6x$

Additional information

Anti-differentiation as the reverse process of differentiation

In this activity, determine the expressions on the right that are anti-derivatives of the expressions on the left.

You can see that $3x^2$, $3x^2 + 2$, and $3x^2 - 7$ are all anti-derivatives of $6x$, differing only by the constant. Hence there is an infinite number of anti-derivatives of $6x$, all with the same gradient function. Therefore you write the anti-derivative of $6x$ as $3x^2 + c$, where c is an arbitrary constant, also known as the *constant of integration*. The notation you use to indicate the process of anti-differentiation is $\int 6x\,dx = 3x^2 + c$, where $c \in \mathbb{R}$. This is called the *indefinite integral* of $6x$ with respect to x.

Note that a constant coefficient can be written before or after the integral sign without changing the value of the integral; for example, $\int 6x\,dx = 6\int x\,dx$.

$\int f(x)dx$ is the indefinite integral or primitive of $f(x)$, and it indicates that you are finding the anti-derivative of the expression $f(x)$ with respect to x. The dx indicates that x is the variable with respect to which the anti-differentiation takes place.

Hence $\int f(x)dx = F(x) + c$

also, $\int f'(x)dx = f(x) + c$

or $\int \frac{dy}{dx}dx = y + c$

The derivative of the integral gives the original function, but the integral of the derivative does not (because of any constant term that may be present, c).

The two useful results in the box below can be used to simplify anti-differentiation. The first one allows placement of a constant coefficient either inside or outside the integral sign. The second one allows for anti-differentiation term by term of a series of terms.

Rules for indefinite integrals:

$$\int k\,f(x)dx = k\int f(x)dx$$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Additional information

Identify the components of an indefinite integral

This activity provides an explanation of the different components of an indefinite integral.

1 Use differentiation to determine an anti-derivative

Differentiate $f(x) = 4x^3 + 3x^2$ and hence determine the anti-derivative of $12x^2 + 6x$.

THINKING	WORKING
1 Determine the derivative.	$f'(x) = 3 \times 4x^{3-1} + 2 \times 3x^{2-1}$ $= 12x^2 + 6x$
2 Rewrite in the form $\int f'(x)dx = f(x) + c$.	$\int (12x^2 + 6x)dx = 4x^3 + 3x^2 + c$

2 Use differentiation to determine an anti-derivative with adjustment of a constant coefficient

Differentiate each of the following and hence determine the required anti-derivative.

(a) Differentiate $f(x) = 6x^4 - 2x^3$ and hence determine the anti-derivative of $4x^3 - x^2$.

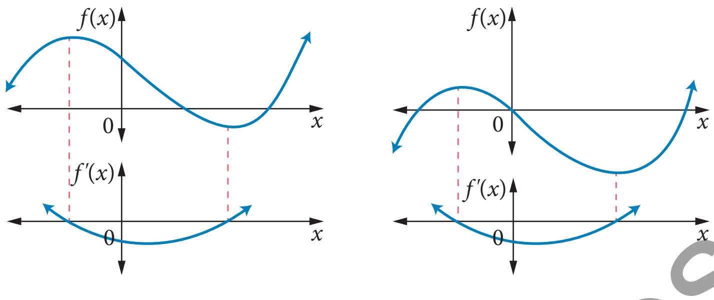
THINKING	WORKING
1 Determine the derivative.	$f'(x) = 24x^3 - 6x^2$
2 Express the derivative in terms of the required anti-derivative.	$f'(x) = 6(4x^3 - x^2)$
3 Express the anti-derivative required in terms of the derivative found where $\int f'(x)dx = f(x) + c$.	$\int (4x^3 - x^2)dx = \frac{1}{6} \times 6\int (4x^3 - x^2)dx$ $= \frac{1}{6}(6x^4 - 2x^3 + c_1)$
4 Express the anti-derivative in the required form.	$= x^4 - \frac{1}{3}x^3 + c$ where $c = \frac{c_1}{6}$

(b) Differentiate $f(x) = (3x - 2)^3$ and hence determine the anti-derivative of $\int 27(3x - 2)^2\, dx$.

1 Differentiate the function.	$f'(x) = 3(3x - 2)^2 \times 3$ $= 9(3x - 2)^2$
2 Express the derivative in terms of the required anti-derivative.	$f'(x) = \frac{1}{3} \times 27(3x - 2)^2$
3 Express the anti-derivative required in terms of the derivative found where $\int f'(x)dx = f(x) + c$.	$\frac{1}{3}\int 27(3x-2)^2\, dx = \int 9(3x-2)^2\, dx$ $\int 27(3x-2)^2\, dx = 3 \times \int 9(3x-2)^2\, dx$ $= 3 \times \left[(3x-2)^3 c_1\right]$
4 Express the anti-derivative in the required form.	$= 3(3x-2)^3 + c$, where $c = 3c_1$

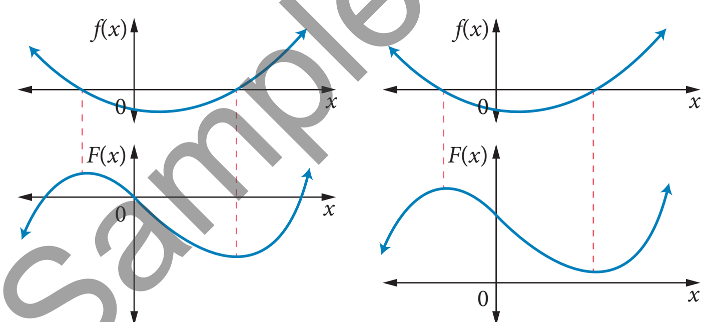
Graphing anti-derivatives

You will recall that the graph of a derivative function $y = f'(x)$ gives the value of the gradient of the original function $y = f(x)$ for each point on the graph. The graphs of each of the two functions below result in the same derivative graph because these functions have the same gradient for every value of x in the domain. You can obtain either of the graphs of $y = f(x)$ from the other by a simple translation up or down.

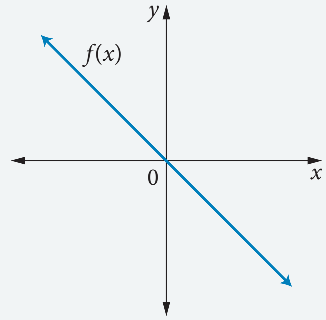
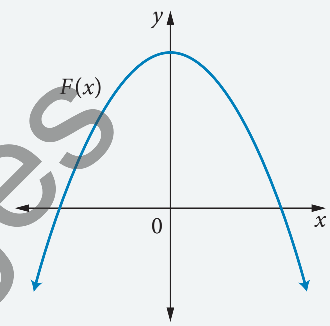


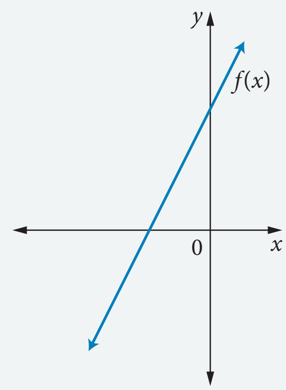
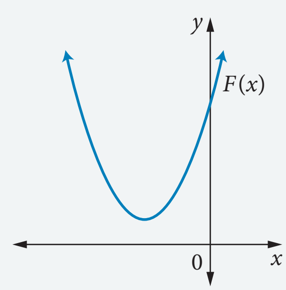
Where the function has a maximum or minimum, the graph of the derivative has an x-intercept, i.e. the function has a gradient of zero. Also, where the function has a positive gradient, the derivative graph is positive, and where the function has a negative gradient, the derivative graph is negative.

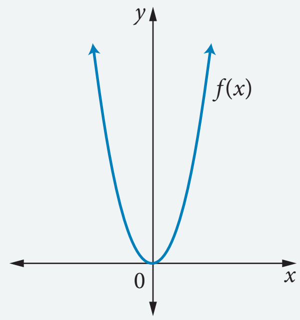
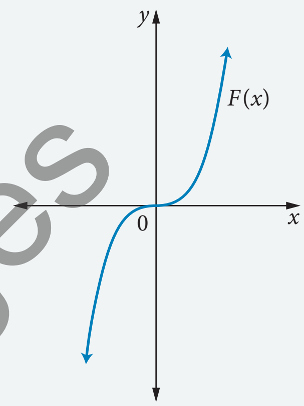
Finding an anti-derivative graph $y = F(x)$, given the graph of a function $y = f(x)$, is the reverse process of finding a derivative graph. You can look at the values of $f(x)$ and create a corresponding gradient for $F(x)$. From the graphs above, it is clear there will not be a unique result, so possible anti-derivative graphs are drawn.

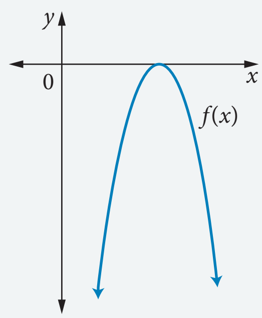
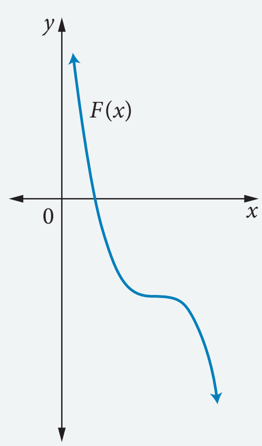


In this case, where the function has an x-intercept, the graph of the anti-derivative has a maximum or minimum. Where the function is positive, the graph of the anti-derivative is increasing, and where the function is negative, the graph of the anti-derivative is decreasing. Wherever you position the possible $y = F(x)$ graph, it belongs to the family of anti-derivative graphs for $y = f(x)$ obtained by translating the graph in a direction parallel to the y-axis. It is possible to determine the nature of the turning point on the graph of $y = F(x)$ by examining the sign of the original function around the x-intercept. Consider the graph above, in which $f(x)$ indicates the gradient of $F(x)$.

Maximum turning point				
Sign of $f(x)$	+	0	–	
Description of $f(x)$	Above the x-axis	x-intercept	Below the x-axis	
Shape of $F(x)$	/	–	\	
Description of $F(x)$	Increasing	Stationary point	Decreasing	

Minimum turning point				
Sign of $f(x)$	–	0	+	
Description of $f(x)$	Below the x-axis	x-intercept	Above the x-axis	
Shape of $F(x)$	\	–	/	
Description of $F(x)$	Decreasing	Stationary point	Increasing	

A stationary point of inflection where $f(x)$ is above the x-axis				
Sign of $f(x)$	+	0	+	
Description of $f(x)$	Above the x-axis	x-intercept	Above the x-axis	
Shape of $F(x)$	/	–	/	
Description of $F(x)$	Increasing	Stationary point	Increasing	

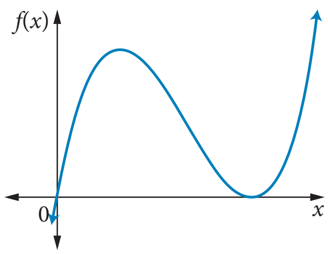
A stationary point of inflection where $f(x)$ is below the x-axis				
Sign of $f(x)$	–	0	–	
Description of $f(x)$	Below the x-axis	x-intercept	Below the x-axis	
Shape of $F(x)$	\	–	\	
Description of $F(x)$	Decreasing	Stationary point	Decreasing	

If the graph of the original function has a discontinuity, such as a hyperbola, you can deduce a possible anti-derivative graph for each part of the domain separately. The graph of $F(x)$ will have asymptotes at the same x-values as $f(x)$; more explicitly if a function has a vertical asymptote, then both the derivative and the anti-derivative are undefined for the value of x at the asymptote.

3 Sketch anti-derivatives

Sketch a possible graph of the anti-derivative of each of the following functions.

(a)



THINKING

1 Consider the x-intercepts of $f(x)$ and the corresponding features of $F(x)$.

WORKING

Near the origin, the gradient follows:

$f(x)$	–	0	+
$F(x)$	\	–	/

A minimum turning point on the graph of $F(x)$.

Near the minimum turning point, the gradient follows:

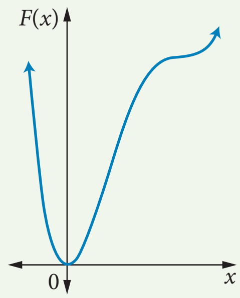
$f(x)$	+	0	+
$F(x)$	/	–	/

A stationary point of inflection exists on the graph of $F(x)$.

2 Consider the other features.

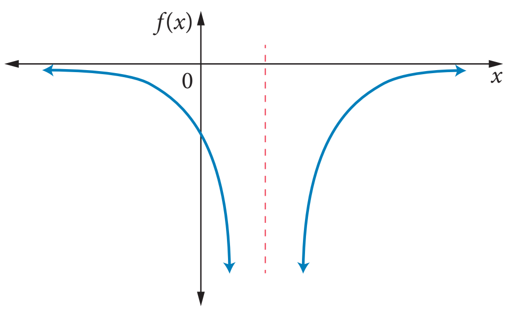
The maximum turning point indicates that the gradient increases to a maximum positive value and then starts to decrease (still positive). There is a point of inflection.

3 Sketch a possible graph of the anti-derivative.



This graph has been drawn passing through the origin, but any vertical translation of the shape is correct.

(b)



1 Consider the asymptotes.

The vertical asymptote will remain in the same position.

The horizontal asymptote indicates that the gradient approaches zero as $x \rightarrow \pm\infty$, but it might not be at $y = 0$.

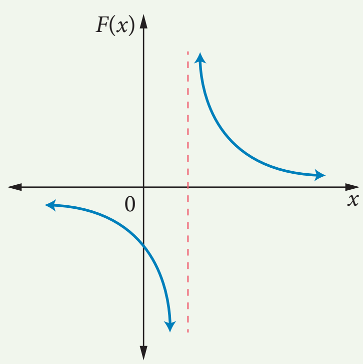
2 Consider the other features.

The derivative (gradient) is always negative.

Left-hand branch: the gradient is close to zero, then becomes increasingly negative.

Right-hand branch: The gradient is negative and becomes less negative, approaching zero.

3 Sketch a possible graph of the anti-derivative.



Any vertical translation of this graph is correct.

EXERCISE 4.1 Introduction to anti-differentiation

Worked Example

1 **1** Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = 2x^3 - 4x$, hence determine the anti-derivative of $6x^2 - 4$.

(b) $f(x) = 3x^2 - 2\sqrt{x}$, hence determine the anti-derivative of $6x - \dfrac{1}{\sqrt{x}}$.

(c) $f(x) = 5x^2 + 2x - 6$, hence determine the anti-derivative of $10x + 2$.

(d) $f(x) = 5x^2 - \dfrac{1}{x}$, hence determine $\displaystyle\int\left(10x + \frac{1}{x^2}\right)dx$.

2 **2** Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = 6x^3 - 10x$, hence determine the anti-derivative of $9x^2 - 5$.

(b) $f(x) = 5x^3 - x^5$, hence determine the anti-derivative of $3x^2 - x^4$.

(c) $f(x) = 3x^4 - 5x^3$, hence determine the anti-derivative of $4x^3 - 5x^2$.

(d) $f(x) = x^2 - \dfrac{1}{x}$, hence determine $\displaystyle\int\left(4x + \frac{2}{x^2}\right)dx$.

3 Differentiate each of the following functions and use the results to determine the given anti-derivatives.

(a) $f(x) = (3x-2)^5$, hence determine $\displaystyle\int 15(3x-2)^4\,dx$.

(b) $f(x) = (6x-1)^3$, hence determine $\displaystyle\int 36(6x-1)^2\,dx$.

4 Differentiate each of the following trigonometric functions and use the results to determine the given anti-derivatives.

(a) $f(x) = \sin(2x)$, hence determine $\displaystyle\int \cos(2x)\,dx$.

(b) $f(x) = 3\cos^3(x)$, hence determine $\displaystyle\int 27\cos^2(x)\sin(x)\,dx$.

(c) $f(x) = \sin^2(3x) - \cos^2(3x)$, hence determine $\displaystyle\int \sin(3x)\cos(3x)\,dx$.

5 Differentiate each of the following exponential functions and use the results to determine the given anti-derivatives.

(a) $f(x) = e^x$, hence determine $\displaystyle\int e^x\,dx$.

(b) $f(x) = e^{kx}$, hence determine $\displaystyle\int e^{kx}\,dx$.

(c) $f(x) = 3e^{5x}$, hence determine $\displaystyle\int 30e^{5x}\,dx$.

6 Differentiate each of the following logarithmic functions and use the results to determine the given anti-derivatives.

(a) $f(x) = \log_e(7x)$, hence determine $\displaystyle\int \frac{1}{2x}\,dx$.

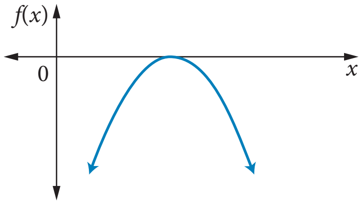
(b) $f(x) = \log_e(2x+1)$, hence determine $\displaystyle\int \frac{1}{2x+1}\,dx$.

Worked Example

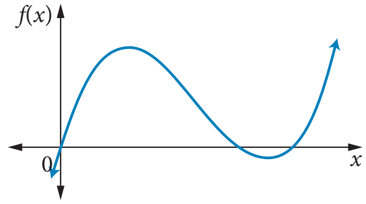
3

7 Sketch the graph of a possible anti-derivative function for each of the given functions, matching key values.

(a)

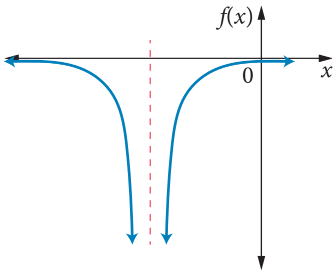


(b)

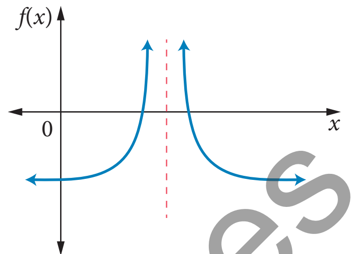


8 Sketch the graph of a possible anti-derivative function for each of the given functions, matching key values.

(a)



(b)



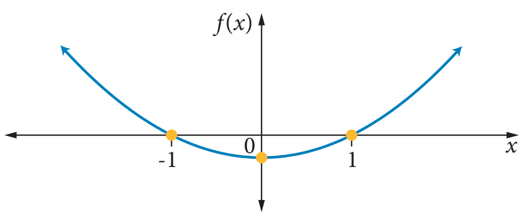
9 If the derivative of $x^2 \sin(2x)$ is $2x(x\cos(2x) + \sin(2x))$, then $\int\left(x^2 \cos(2x) + x\sin(2x)\right)dx$ is equal to which one of the following?

A $x^2 \sin(2x) + c$ **B** $2x(x\cos(2x) + \sin(2x)) + c$ **C** $\frac{1}{2}x^2 \sin(2x) + c$ **D** $2x^2 \sin(2x) + c$

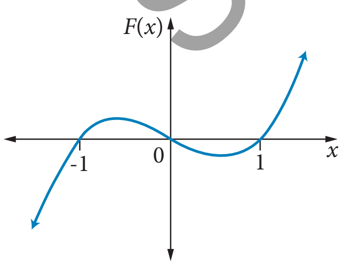
10 Which one of the following integrals cannot be determined by differentiating x^3e^x?

A $\int x^2e^x(x+3)\,dx$ **B** $\int e^x(x+3)\,dx$ **C** $\int \frac{1}{2}e^x\left(x^3+3x^2\right)dx$ **D** $\int 2x^3e^x + 6x^2e^x\,dx$

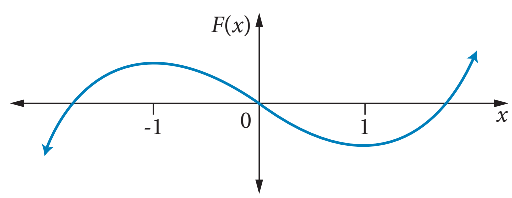
11 Which one of the following is a possible anti-derivative graph for the graph of $f(x)$ shown?



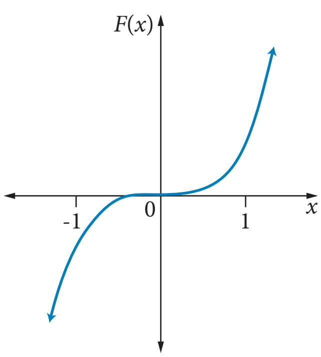
A



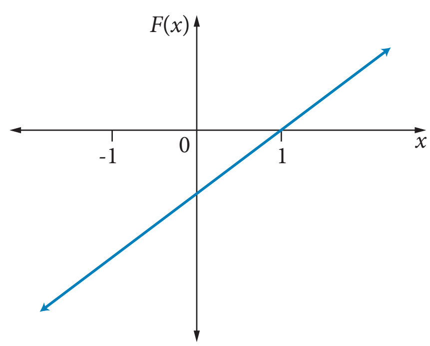
B



C



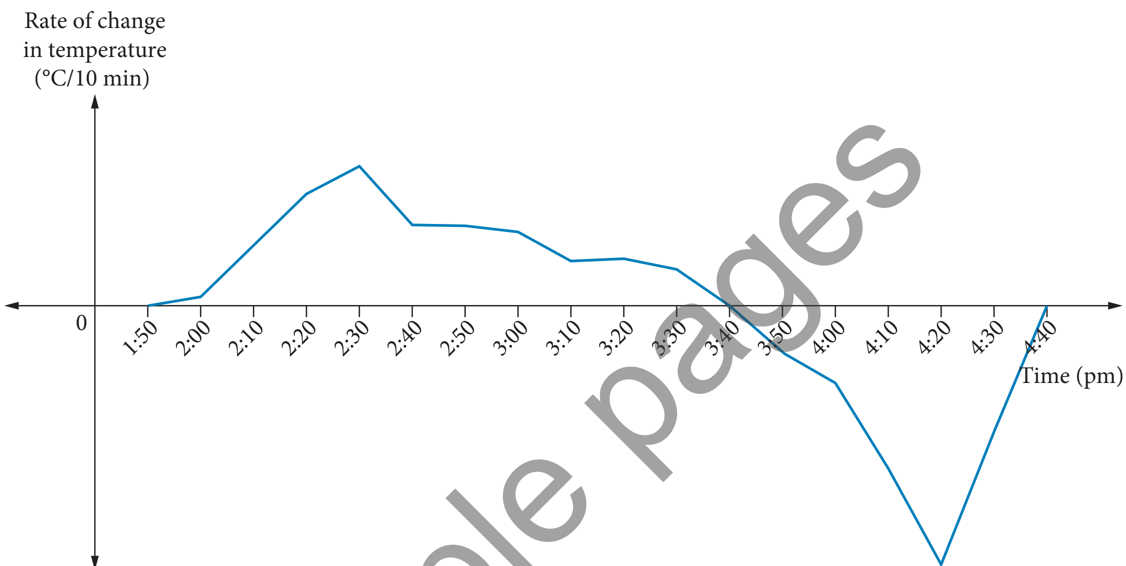
D



12 Verify that the derivative of $x^2(2x-1)^3$ can be expressed as $2x(5x-1)(2x-1)^2$. Hence, determine the value of the constant of integration c, given that $\int x(5x-1)(2x-1)^2\,dx = 20$ at $x = 2$.

13 Differentiate $3(2x-1)^4$ and hence determine the equation of the curve $y = f(x)$ with a gradient function of $3(2x-1)^3$ that passes through the point $(1.5, 8)$.

14 A child began to feel unwell around 1:30 pm on Sunday, and by 1:50 pm she was very hot and feverish. The child's temperature was taken at 10-minute intervals for the next few hours. The graph shows the rate of change of temperature over time.



(a) At what approximate time was the child's temperature rising most rapidly?

(b) Between what times was the temperature rising?

(c) Describe what was happening to the temperature at approximately 3:40 pm.

(d) Sketch a possible graph of the child's temperature over the same period of time.

Anti-differentiation of power functions 4.2

In the previous section, you were introduced to anti-differentiation as the reverse operation of differentiation, noting that the indefinite integral has an arbitrary constant added. In this section, you will learn how to determine the anti-derivative (indefinite integral) of a power function and use given conditions to determine a particular value of the anti-derivative.

Consider the following derivatives:

$$\frac{d\left(x^2\right)}{dx} = 2x$$
$$\int 2x\,dx = x^2 + c_1$$
$$2\int x\,dx = x^2 + c_1$$
$$\int x\,dx = \frac{1}{2}\left[x^2 + c_1\right]$$
$$\int x\,dx = \frac{1}{2}x^2 + c$$

where $c = \frac{1}{2}c_1$

$$\frac{d\left(x^3\right)}{dx} = 3x^2$$
$$\int 3x^2\,dx = x^3 + c_1$$
$$3\int x^2\,dx = x^3 + c_1$$
$$\int x^2\,dx = \frac{1}{3}\left[x^3 + c_1\right]$$
$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

where $c = \frac{1}{3}c_1$

Additional information

Anti-differentiating powers of x

Practise using the pattern for anti-differentiating expressions of the form x^n.

You can check these by differentiating the result in each case.

Note that for the anti-derivative, the power is increased by one and the result is divided by the new power.

In the general case:

$$\int x^n\,dx = \frac{1}{n+1}x^n + c, \quad n \neq -1$$

As you will have observed in the previous section, a constant coefficient may be written inside the anti-derivative sign or before it.

$$\int ax^n\,dx = a\int x^n\,dx, \quad n \neq -1$$
$$= \frac{ax^{n+1}}{n+1} + c$$

4 Anti-differentiate expressions of the form ax^n

Determine the following anti-derivatives.

(a) $\int 3x^4\,dx$

THINKING	WORKING
1 Express in the form $a\int x^n\,dx$.	$\int 3x^4\,dx = 3\int x^4\,dx$
2 Recall the formula.	$a\int x^n\,dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$
3 Substitute the known values.	$= \frac{3x^5}{5} + c$
4 Interpret the answer.	The anti-derivative of $3x^4$; that is, $\int 3x^4\,dx$ is equal to $\frac{3x^5}{5} + c$.

(b) $\int 4x\sqrt{x}\,dx$

1 Express in the form $a\int x^n dx$.	$\int 4x\sqrt{x}\,dx = 4\int x^{\frac{3}{2}} dx$
2 Recall the formula.	$a\int x^n dx = \dfrac{ax^{n+1}}{n+1} + c, \quad n \neq -1$
3 Substitute the known values and simplify as needed.	$4\int x^{\frac{3}{2}} dx = \dfrac{4x^{\frac{5}{2}}}{\frac{5}{2}} + c$ $= \dfrac{8x^{\frac{5}{2}}}{5} + c$
4 Interpret the answer.	The anti-derivative of $4x\sqrt{x}$; that is, $\int 4x\sqrt{x}\,dx$ is equal to $\dfrac{8x^{\frac{5}{2}}}{5} + c$.

You will note that if $f(x) = 2x$, then $f'(x) = 2$, hence $\int 2\,dx = 2x + c$. Writing $\int 2\,dx$ as $\int 2x^0 dx$ and applying the rule for anti-differentiating x^n, you obtain:

$$2\int x^0 dx = 2 \times \frac{x^1}{1} + c$$
$$= 2x + c$$

$$\int a\,dx = ax + c$$

The anti-derivative of a sum or difference of a series of terms is equal to the sum or difference of the anti-derivative of each individual term.

$$\int \left(f(x) \pm g(x)\right) dx = \int f(x)dx \pm \int g(x)dx$$

5 Anti-differentiate term by term

Determine $\int \left(3x^2 - 3\sqrt{x} + 4\right) dx$.

THINKING	WORKING
1 Express terms in the form $a\int x^n dx$.	$\int \left(3x^2 - 3\sqrt{x} + 4\right) dx = 3\int x^2 dx - 3\int x^{\frac{1}{2}} dx + \int 4\,dx$ This step can be omitted, and the anti-differentiation can be done under the one integral sign.
2 Use the rule for anti-differentiating powers of x. Since c represents an arbitrary constant, you only need to add c once at the end.	$\int \left(3x^2 - 3\sqrt{x} + 4\right) dx = 3\left(\dfrac{x^3}{3}\right) - 3\left(\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + 4x + c$
3 Simplify the expression on the right-hand side of the equation.	$= x^3 - 3 \times \dfrac{2}{3} x^{\frac{3}{2}} + 4x + c$
4 Express the anti-differentiated expression.	$\int \left(3x^2 - 3\sqrt{x} + 4\right) dx = x^3 - 2x^{\frac{3}{2}} + 4x + c$

Notation

The following notations are used at various times to indicate the process of anti-differentiation.

$$\int f'(x)dx = f(x)+c$$

$$\int \frac{dy}{dx}dx = y+c$$

$$\int f(x)dx = F(x)+c$$

6 Simplify before anti-differentiating

Determine the anti-derivative of $\dfrac{(3x^2-2)(x^2+1)}{x^2}$.

THINKING	WORKING
1 Expand the brackets and collect like terms.	$\int \frac{(3x^2-2)(x^2+1)}{x^2}dx = \int \frac{3x^4+x^2-2}{x^2}dx$
2 Rewrite each term with the denominator and express each one in the form ax^n.	$= \int \left(\frac{3x^4}{x^2}+\frac{x^2}{x^2}-\frac{2}{x^2}\right)dx$ $= \int (3x^2+1-2x^{-2})dx$
3 Anti-differentiate each term using the rule and introduce the constant of integration.	$= 3\times\frac{x^3}{3}+x-2\times\frac{x^{-1}}{-1}+c$
4 Simplify, expressing with positive powers, and write down the answer.	$= x^3+x+2x^{-1}+c$
5 Express the anti-differentiated expression.	$\int \frac{(3x^2-2)(x^2+1)}{x^2}dx = x^3+x+\frac{2}{x}+c$

Applications of anti-differentiation

Given a rate of change, anti-differentiating can lead you to information about the quantity whose rate you are measuring. For example, velocity is a measure of the rate of change of position. In other words, the velocity v may be expressed as $v = \frac{dx}{dt}$, where x is the position of a particle at time t. Anti-differentiating with respect to t gives you an expression for the position.

7 Apply anti-differentiation

Determine an expression for the position x of an object if its velocity v is modelled by the function $v = \frac{3}{t^2} - 5$ at time t, for $t \in [2,5]$.

THINKING	WORKING
1 Express as an anti-derivative.	$x = \int v\,dt$ $= \int \left(\frac{3}{t^2}-5\right)dt$

2 Write terms in the form $a\int x^n dx$.	$\int\left(\frac{3}{t^2}-5\right)dt=\int\left(3t^{-2}-5t^0\right)dt$
3 Use the rule to anti-differentiate each term, introduce the constant of integration and simplify the expression.	$=3\times\frac{t^{-1}}{-1}-\frac{5t^1}{1}+c$ $=-\frac{3}{t}-5t+c$
4 Interpret the answer.	The position equation is given by $x=-\frac{3}{t}-5t+c$.

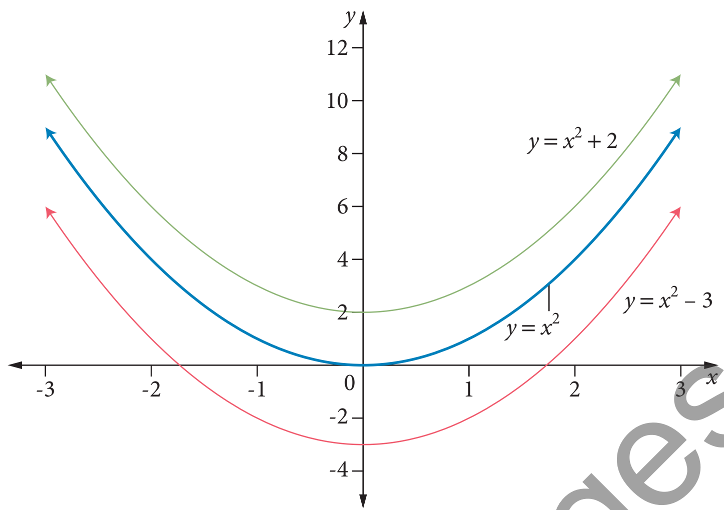
8 Determine the value of the constant of integration

If $\frac{dy}{dx}=\left(4-\frac{3}{x^2}\right)^2$, determine y in terms of x if $y=13$ when $x=1$.

THINKING	WORKING
1 Expand the expression and write each term in the form $a\int x^n dx$.	$\left(4-\frac{3}{x^2}\right)^2=16-\frac{24}{x^2}+\frac{9}{x^4}$ $=16-24x^{-2}+9x^{-4}$
2 Express y as an anti-derivative of $\frac{dy}{dx}$.	$y=\int\frac{dy}{dx}dx$ $=\int\left(16-24x^{-2}+9x^{-4}\right)dx$
3 Anti-differentiate the expression, introduce the constant of integration and simplify.	$=16x-24\times\frac{x^{-1}}{-1}+9\times\frac{x^{-3}}{-3}+c$ $=16x+\frac{24}{x}-\frac{3}{x^3}+c$
4 Use the given conditions to determine the value of the constant.	When $x=1$, $y=13$: $13=16(1)+\frac{24}{1}-\frac{3}{1^3}+c$ $13=16+24-3+c$ $c=-24$
5 Interpret the answer.	The anti-derivative of $\frac{dy}{dx}=\left(4-\frac{3}{x^2}\right)^2$ that includes the point $(1,13)$ is $y=16x+\frac{24}{x}-\frac{3}{x^3}-24$.

Families of curves

If $f'(x) = 2x$, then $f(x) = x^2 + c$. A set of graphs for various values of the arbitrary constant c represents a family of curves, all of which have the same gradient function $2x$.



Each graph represents a translation of $y = x^2$ parallel to the y-axis. At any x-value, the curves will have the same gradient.

Additional information

Changing the value of the constant

Use the slider and observe the gradient of the tangents when the graph is translated up or down, representing different values of the constant.

EXERCISE 4.2

Anti-differentiation of power functions

Worked Example

1 Determine each of the following anti-derivatives. **4**

(a) $\int x^6 \, dx$ (b) $\int 12x^3 \, dx$ (c) $\int 4x^2 \, dx$ (d) $\int \frac{1}{3}x^5 \, dx$

(e) $\int 3\sqrt{x} \, dx$ (f) $\int 2x^{\frac{2}{3}} \, dx$ (g) $\int \frac{3}{4}x^{\frac{2}{5}} \, dx$ (h) $\int 3x^{-2} \, dx$

2 Determine each of the following anti-derivatives. **5**

(a) $\int \left(6x^2 - x + 2\right) dx$ (b) $\int \left(x^3 - 3x^2 + 2x\right) dx$ (c) $\int \left(6x^2 + 2\sqrt{x}\right) dx$

(d) $\int \left(4 - \frac{x^2}{2}\right) dx$ (e) $\int \left(4x - \frac{1}{\sqrt{x}} + x^3\right) dx$ (f) $\int \left(7 + \frac{2}{3w^2} + w^{\frac{2}{3}}\right) dw$

3 Determine each of the following anti-derivatives. **6**

(a) $\int \frac{x^2 + x}{x} \, dx$ (b) $\int \frac{x^3 - 2x^2}{x^2} \, dx$ (c) $\int \frac{x^4 - 2x^2 + 3}{x^2} \, dx$

(d) $\int (x-1)\left(x^2 + 2\right) dx$ (e) $\int \frac{(x-2)(x+2)}{x^2} \, dx$ (f) $\int \frac{\left(x^2 - 3\right)\left(x^2 + 4\right)}{x^2} \, dx$

(g) $\int \frac{x^2 + x}{\sqrt{x}} \, dx$ (h) $\int \sqrt{x}\,(x - 3) \, dx$

Worked Example

7 **4** The velocity v of a particle is the rate of change of its position x at time t, so $v = \frac{dx}{dt}$. If $v = 3t^2 - 4t + 18$, determine an expression for the position of the particle at time t.

8 **5** If the particle in question **4** is at a position $x = 2$ when $t = 0$, determine the value of each of the following.

(a) the constant of integration, c

(b) the position of the particle when $t = 3$

6 Determine $f(z)$, given:

(a) $f'(z) = z - 6z^2$

(b) $f'(z) = (z-2)(z+4)$

(c) $f'(z) = \dfrac{3z^5 - 4z}{z^3}$

(d) $f'(z) = \dfrac{-2}{z^2}$

7 For each of the following, determine y in terms of x for the given conditions.

(a) $y = \int \left(6x^2 - 2x\right) dx$, and $y = -10$ when $x = 2$

(b) $\dfrac{3}{x^2} - \dfrac{4}{x^3}$, and $y = 1$ when $x = 4$

8 The gradient of a curve at any point is given by $f'(x) = \left(1 + \sqrt{x}\right)^2$. Determine the equation of the curve if it passes through the point $\left(1, -\dfrac{1}{6}\right)$.

9 Determine an anti-derivative of each of the following.

(a) $\int \left(3 - \dfrac{x}{2}\right)^5 dx$

(b) $\int \sqrt{x}\,(4-x)^2\,dx$

10 For each of the following, express y in terms of x.

(a) $\dfrac{dy}{dx} = 8x^3 + \dfrac{3}{x^2}$

(b) $\dfrac{dy}{dx} = 2x^{-3} + x^{-\frac{3}{2}}$

11 Differentiate $f(x) = \dfrac{2}{3}\left(1 + x^2\right)^{\frac{3}{2}}$ and use the result to determine $\int 2x\sqrt{1 + x^2}\,dx$.

12 The gradient of a curve varies directly as x^2. If the curve passes through the origin and the point $(3, 18)$, determine each of the following.

(a) the equation of the curve

(b) the value of the gradient at $x = -3$

(c) the value of y at $x = -3$

13 The gradient of a curve is given by $\dfrac{dy}{dx} = 3(2x - 1)^2$.

(a) Determine the equation of the family of curves that have this gradient.

(b) Determine the member of this family of curves that passes through each of the given points.

(i) $(0, 4)$ **(ii)** $(-1, 5)$

(c) Explain why the two curves from part **(b)** have no points of intersection.

14 The position x, velocity v, and acceleration a of a particle at time t are related by: $v = \frac{dx}{dt}$, $a = \frac{dv}{dt}$. If the acceleration of a particle is modelled by $a = 3t^2 - t - 2,\ t \geq 0$, determine the following.

(a) the velocity of the particle at $t = 10$ if the particle starts from rest

(b) the position of the particle at $t = 2$, if $x = \frac{1}{12}$ when $t = 1$

15 A tank of water is emptying at a rate given by $\frac{dV}{dt} = -2.8t$, where V is the volume in cubic metres m^3 at time t seconds.

(a) Determine an expression for the volume of water remaining in the tank in terms of time, if it is empty after 30 seconds.

(b) How long does it take for half of the initial volume to empty out? Give your answer correct to 1 decimal place.

16 Which one of the following is the anti-derivative of $x\left(3x^2 - 1\right)$?

A $\frac{x^2}{2}\left(\frac{3x^3}{3} - x\right) + c$

B $\left(3x^2 - 1\right) \times 1 + x(6x) + c$

C $3x^4 - x + c$

D $\frac{1}{2}x^2\left(\frac{3x^2}{2} - 1\right) + c$

17 Which one of the following statements is correct?

A $\int (x - 3)dx$ represents a family of parabolas with axes of symmetry at $x = 3$.

B $\int (x - 3)dx$ represents a family of parabolas with axes of symmetry at $x = 6$.

C $\int (x - 3)dx$ represents a family of parabolas with two x-intercepts.

D $\int (x - 3)dx$ could be equal to $\frac{x^2}{2} - 3x + 10x^2$.

18 Determine the following anti-derivatives.

(a) $\int \frac{\left(x + \frac{1}{x}\right)^2}{x^2} + 5\,dx$

(b) $\int \frac{x^3 - 2\sqrt{x} + x^{-2}}{x^2}\,dx$

19 Differentiate $f(x) = (x + 3)^{\frac{1}{2}}x^2$ and hence determine $\int \frac{5x^2 + 12x}{2\sqrt{x + 3}}\,dx$.

20 If $f'(x) = px^2 + \frac{2}{x^3}$, determine $f(x)$ if $f(1) = 2$ and $f(-1) = 4$.

21 Differentiate $(3x + 1)^4$ and hence determine the anti-derivative of $(3x + 1)^3$.

22 (a) Differentiate $(ax + b)^{n+1}$ and hence determine the anti-derivative of $\int (ax + b)^n\,dx$.

(b) Use the result found in part (a) to determine the following.

(i) $\int (6x + 1)^4\,dx$

(ii) $\int (3 - 2x)^3\,dx$

(iii) $\int (3x - 2)^{\frac{1}{2}}\,dx$

(iv) $f(x)$ if $f'(x) = 4(7x + 2)^3$ and $f(0) = -\frac{3}{7}$