NEW SENIOR MATHEMATICS

EXTENSION 1

FOR YEARS 11 & 12

THIRD EDITION

J.B. FITZPATRICK

BOB AUS

NSW STAGE 6

INTRODUCTION AND DEDICATION

J.B. Fitzpatrick

It is interesting to wonder whether J. B. Fitzpatrick ('Bernie') realised in 1983 just how popular his book *New Senior Mathematics* would be. That first edition of *New Senior Mathematics* was to remain in print for almost 30 years. It has stood the test of time thanks to the quality, rigour and variety of its questions, its accuracy and its high mathematical standards.

As Fitzpatrick wrote in 1983: 'Mathematics, like many other things, is best learnt by doing. A student begins to appreciate the power of mathematics when he or she has achieved a mastery of basic techniques, not after reading lengthy explanations … The emphasis throughout the book is on the understanding of mathematical concepts' (Introduction, *New Senior Mathematics* 1984).

J. B. Fitzpatrick passed away in 2008. Fitzpatrick was a respected author, teacher and figurehead of mathematics education.

Bob Aus

Bob Aus taught in New South Wales high schools for 40 years, retiring in 2007. During that time Bob taught all courses from Years 7 to 12 up to Level 1 / 4-unit / Extension 2. He has marked HSC examination papers and has been involved in the standards setting process as judge and chief judge for the three Calculus-based courses over four years. He has also completed review work for the NSW Board of Studies and represented NSW at a week-long review and standards setting of the upper level course from each state prior to the development of the Australian National Curriculum for senior students.

Bob spent time as Regional Vocational Education Consultant in the North Coast region and was a Mathematics consultant in the Hunter region. When he retired he was Head Teacher Mathematics at Merewether High School and enjoyed teaching an Extension 2 class with 24 students.

Bob's first publication was in 1983 and he has been involved with writing a range of textbooks and study guides since then, including revising and updating the *New Senior Mathematics* series 2nd edition in 2013.

Bob has presented talks on the three Calculus-based courses throughout the state. He has co-written the Years 6–9 Mathematics syllabus for the Abu Dhabi Education Authority, as well as managing the writing project for support material for this course. He also wrote the Years 10–12 syllabus for their Calculus-based course.

This third edition of *New Senior Mathematics* updates it for the new Stage 6 HSC courses in NSW to be implemented in Year 11, 2019.

NEW SENIOR MATHEMATICS

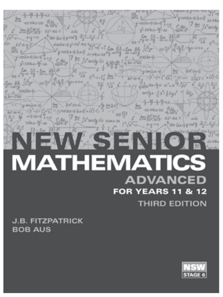
THIRD EDITION

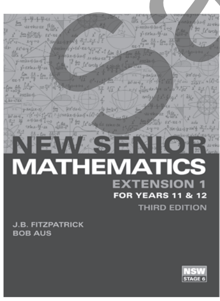
New Senior Mathematics Extension 1 for Years 11 & 12 is part of a new edition of the well-known Mathematics series for New South Wales. The series has been updated to address all requirements of the new Stage 6 syllabus. We have maintained our focus on mathematical rigour and challenging student questions, while providing new opportunities for students to consolidate their understanding of concepts and ideas with the aid of digital resources and activities.

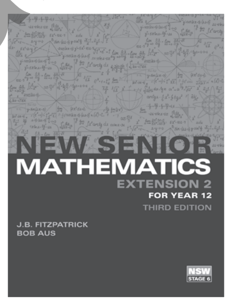
Student Book

The first three chapters of the first student book contain revision material that provides the necessary foundation for the development of senior mathematics concepts. In the new edition you'll also find:

- content built on a rigorous, academic approach that promotes excellence and prepares students for higher education
- a simple, convenient approach with Year 11 and 12 content in one book for Advanced and Extension 1, with colour coding to distinguish year levels
- digital technology activities that promote a deeper understanding, allowing students to make connections, and visualise and manipulate data in real time.

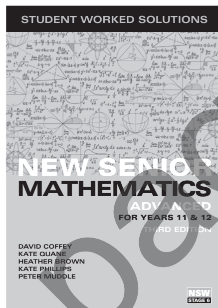


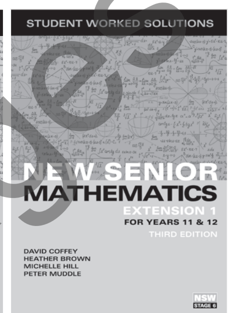


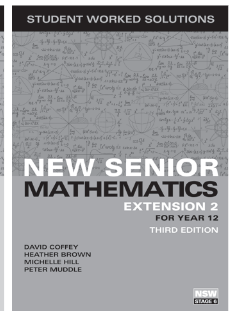


Student Worked Solutions

The *New Senior Mathematics Extension 1 for Years 11 & 12 Student Worked Solutions* contains the fully worked solutions for every second question in *New Senior Mathematics Extension 1 for Years 11 & 12*.







Reader+

Reader+, our next generation eBook, features content and digital activities, with technology such as graphing software and spreadsheets, to help students engage on their devices.

There are also teacher support materials, such as practice exams, question banks, investigation assignments, and fully worked solutions to cover all internal and external assessment items and save you time.

Pearson Places

Pearson

FEATURES OF THE 3RD EDITION STUDENT BOOK/READER+

YEAR LEVELS

Year levels are indicated on each page for easy identification of Year 11 and 12 content.

YEAR 11

YEAR 12

MAKING CONNECTIONS

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their conceptual understanding.

MAKING CONNECTIONS

EXPLORING FURTHER

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skill base. These activities should take approximately 5–15 minutes to complete.

EXPLORING FURTHER

CHAPTER REVIEW

Each chapter contains a comprehensive review of chapter content.

CHAPTER REVIEW

SUMMARY PAGES

A comprehensive course summary is provided at the end of the book.

SUMMARY

CONTENTS

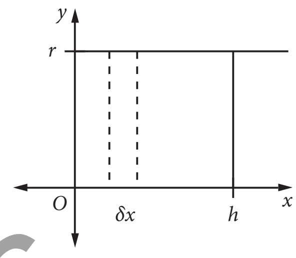
YEAR 12

SYLLABUS REFERENCE

CHAPTER 11
Applications of calculus

11.1 VOLUMES OF SOLIDS OF REVOLUTION

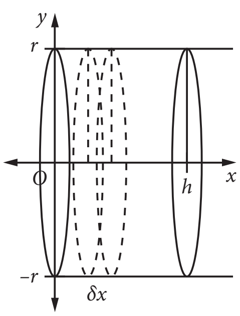
You have seen that the area of a region bounded by a line $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ can be found by adding up the areas of all the rectangles of width δx and height r between $x = 0$ and $x = h$, as δx becomes vanishingly small: $A = \lim_{\delta x \to 0} \sum_{0}^{h} f(x)\delta x$.



This area is given by the definite integral $A = \int_0^h r\,dx$, which is $A = \int_0^h r\,dx = [rx]_0^h = rh$.

You should recognise this as the area of a rectangle of sides r and h.

Consider what happens when the area bounded by $y = r$, the x-axis and the ordinates $x = 0$ and $x = h$ is rotated about the x-axis to form a solid of revolution, as shown in the diagram below to the left. The solid of revolution formed is a cylinder of radius r and height h.

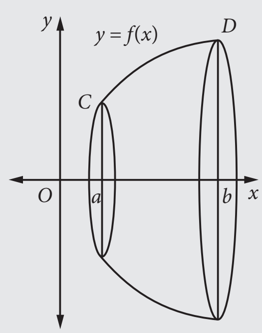


The rectangles of side r and width δx have become circular disks of radius r and thickness δx. The volume of this disk is given by $\Delta V = \pi\left(f(x)\right)^2 \delta x$. Adding all the disks as δx gets smaller gives $V = \lim_{\delta x \to 0} \sum_{0}^{h} \pi\left(f(x)\right)^2 \delta x$, which is given by the definite integral $V = \pi \int_0^h r^2\,dx$.

Thus the volume is $V = \pi \int_0^h r^2\,dx = \pi\left[r^2 x\right]_0^h = \pi r^2 h$, which you should recognise as the volume of a cylinder of radius r and height h.

When the arc CD of the curve $y = f(x)$ on the interval $a \le x \le b$ is rotated about the x-axis, the volume of the solid of revolution formed is given by:

$$V = \pi \int_a^b \left(f(x)\right)^2 dx \quad \textbf{or} \quad V = \pi \int_a^b y^2\,dx$$

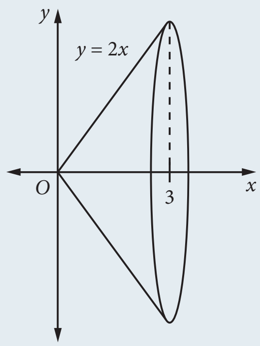


Example 1

Calculate the volume of the solid formed when the portion of the line $y = 2x$ between $x = 0$ and $x = 3$ is rotated about the x-axis. What is the name of the kind of solid formed?

Solution

Draw a diagram:

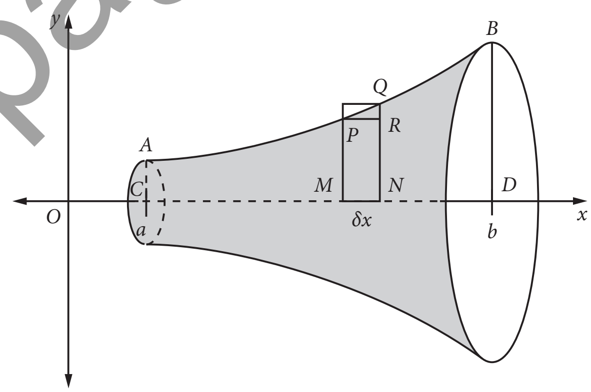


$$
\begin{aligned}
\text{Volume} &= \pi \int_a^b y^2\,dx \\
&= \pi \int_0^3 (2x)^2\,dx \\
&= 4\pi \int_0^3 x^2\,dx \\
&= 4\pi \left[\frac{x^3}{3}\right]_0^3 \\
&= 4\pi(9 - 0) \\
&= 36\pi \text{ units}^3
\end{aligned}
$$

The solid is a right circular cone of base radius 6 and height 3.

Volumes of solids of revolution—formal development

Consider a continuous function f in the interval $a \le x \le b$. If the plane section $ABDC$ is rotated about the x-axis then a solid is generated with circular vertical cross-sections, as shown in the diagram on the right. This solid is called a **solid of revolution**.



$P(x, y)$ is a point on the curve $y = f(x)$ and $Q(x + \delta x, y + \delta y)$ is a point close to P. The ordinate PM describes a circle of area πy^2 and QN describes a circle of area $\pi(y + \delta y)^2$.

The typical lower rectangle $PRNM$ describes a cylinder of volume $\pi y^2\,\delta x$ and the typical upper rectangle describes a cylinder of volume $\pi(y + \delta y)^2\,\delta x$. If a typical layer $PQNM$ describes a solid of volume δV, then:

$$\pi y^2\,\delta x < \delta V < \pi(y + \delta y)^2\,\delta x$$

Thus: $$\sum_a^b \pi y^2\,\delta x < V < \sum_a^b \pi(y + \delta y)^2\,\delta x$$

As $\delta x \to 0$:

$$
\begin{aligned}
V &= \lim_{\delta x \to 0} \sum_a^b \pi y^2\,\delta x \\
&= \int_a^b \pi y^2\,dx \\
&= \pi \int_a^b y^2\,dx
\end{aligned}
$$

Hence, volume of a solid of revolution:

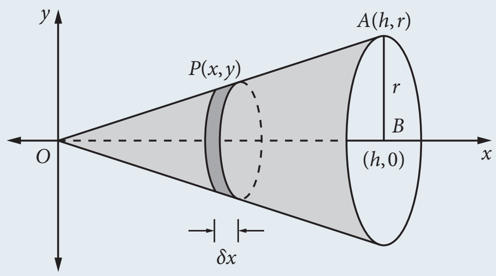
$V = \pi \int_a^b y^2\,dx$ where $y = f(x)$

Example 2

Find the volume of a right circular cone of height h and base radius r.

Solution

The cone can be considered as a solid of revolution generated by rotating the right-angled triangle OAB about the x-axis.



The equation of OA is $y = \frac{rx}{h}$.

$$V = \pi \int_a^b y^2\,dx$$

$$V = \pi \int_0^h \frac{r^2x^2}{h^2}\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi r^2}{h^2} \times \frac{h^3}{3}$$

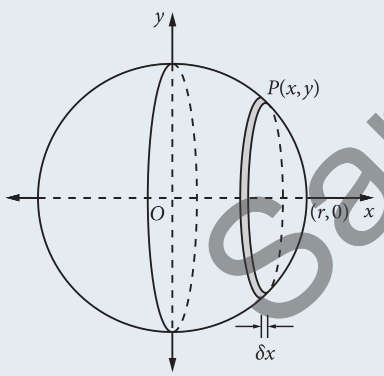
$$= \frac{1}{3}\pi r^2 h$$

Example 3

Find the volume of a sphere of radius r.

Solution

The volume of a sphere can be considered as the volume generated by rotating the semicircle defined by $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis.



Hence:

$$V = \pi \int_{-r}^{r} y^2\,dx \quad \text{where } y = \sqrt{r^2 - x^2}$$

$$= \pi \int_{-r}^{r} \left(r^2 - x^2\right)dx \quad \text{because } y^2 = r^2 - x^2$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right)$$

$$= \frac{4}{3}\pi r^3$$

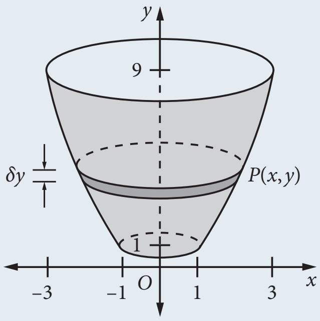
Example **3**, above, proves the formula for the volume of the sphere—a formula that you have used for many years. The formula for the area of a circle $A = \pi r^2$ can similarly be proved using calculus.

Example 4

The part of the parabola $y = x^2$ between $x = 1$ and $x = 3$ is rotated about the y-axis. Calculate the volume generated.

Solution

$y = x^2$: $\quad x = 1, y = 1; x = 3, y = 9$



$V = \pi\int_1^9 x^2\,dy$ where $x^2 = y$

$V = \pi\int_1^9 y\,dy$

$= \pi\left[\frac{y^2}{2}\right]_1^9$

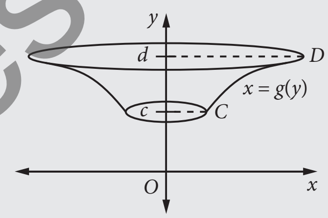
$= \pi\left(\frac{81}{2} - \frac{1}{2}\right)$

$= 40\pi$ units^3

Rotating about the *y*-axis

When the arc CD of the curve $x = g(y)$ on the interval $c \le y \le d$ is rotated about the y-axis, the volume of the solid of revolution formed is given by:

$$V = \pi\int_c^d \left(g(y)\right)^2 dy \quad \textbf{or} \quad V = \pi\int_c^d x^2\,dy$$

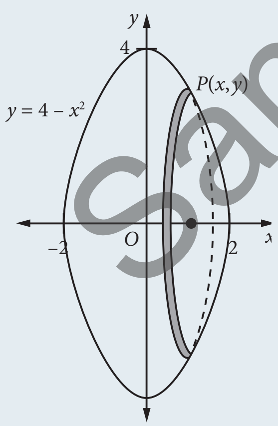


Example 5

Find the volume of the solid formed when the area bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

Solution

(a) Rotate about x-axis:



$V = \pi\int_{-2}^2 y^2\,dx$ where $y = 4 - x^2$

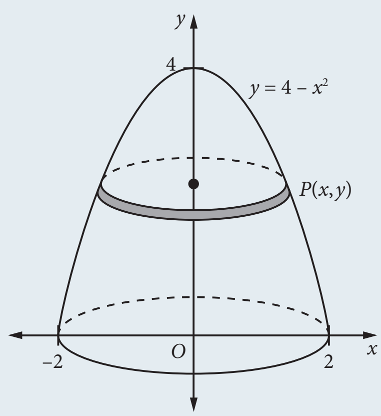
$= \pi\int_{-2}^2 \left(16 - 8x^2 + x^4\right)dx$

$= \pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^2$

$= \pi\left(\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right)$

$= \frac{512\pi}{15}$ units^3

(b) Rotate about y-axis:



$V = \pi\int_0^4 x^2\,dx$ where $x^2 = 4 - y$

$= \pi\int_0^4 (4 - y)\,dx$

$= \pi\left[4y - \frac{y^2}{2}\right]_0^4$

$= \pi\left((16 - 8) - 0\right)$

$= 8\pi$ units^3

Example 6

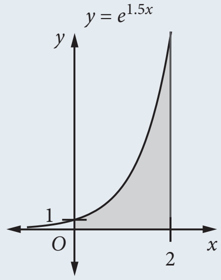
Calculate: **(a)** the area bounded by the curve $y = e^{1.5x}$, the coordinate axes and the line $x = 2$

(b) the volume obtained by rotating this area about the x-axis.

Solution

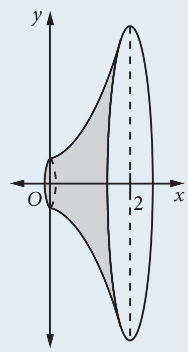
(a) $y = e^{1.5x}$, $y = 0$, $x = 2$

$$\text{Area} = \int_0^2 e^{1.5x}\,dx$$
$$= \left[\frac{2}{3}e^{1.5x}\right]_0^2$$
$$= \frac{2}{3}\left(e^3 - e^0\right)$$
$$= \frac{2\left(e^3 - 1\right)}{3} \approx 12.72 \text{ units}^2$$



(b) $\text{Volume} = \pi\int_0^2 y^2\,dx$ where $y = e^{1.5x}$.

$$= \pi\int_0^2 e^{3x}\,dx$$
$$= \frac{\pi}{3}\left[e^{3x}\right]_0^2$$
$$= \frac{\pi}{3}\left(e^6 - e^0\right)$$
$$= \frac{\pi\left(e^6 - 1\right)}{3}$$
$$\approx 421.4 \text{ units}^3$$



Example 7

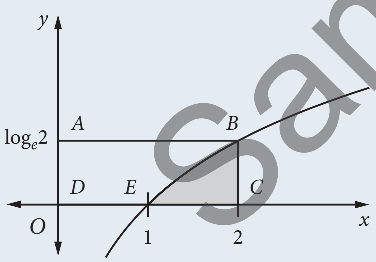
Find: **(a)** the area bounded by the curve $y = \log_e x$, the x-axis and the ordinate $x = 2$

(b) the volume of the solid of revolution formed by rotating the area bounded by the curve $y = \log_e x$, the coordinate axes and the line $y = \log_e 2$ about the y-axis.

Solution

(a) $\text{Area} = \int_1^2 \log_e x\,dx$

Instead of trying to evaluate this integral directly, draw a diagram.



This problem requires the area of the shaded region BCE. It can be obtained by finding the area of the rectangle $ABCD$ and subtracting the area $ABED$.

Because $y = \log_e x$, you can write $x = e^y$.

At $x = 2$, $y = \log_e 2$: $\text{Area } ABED = \int_0^{\log_e 2} e^y\,dy$

$$= \left[e^y\right]_0^{\log_e 2}$$
$$= e^{\log_e 2} - e^0$$
$$= 2 - 1$$
$$= 1$$

$\text{Area } ABCD = 2\log_e 2$

$\therefore \text{Area } BCE = 2\log_e 2 - 1$

$\approx 0.386 \text{ units}^2$

(b) $\text{Volume} = \pi\int_0^{\log_e 2} x^2\,dy \qquad \text{where } x = e^y$

$$= \pi\int_0^{\log_e 2} e^{2y}\,dy$$

$$= \frac{\pi}{2}\left[e^{2y}\right]_0^{\log_e 2}$$

$$= \frac{\pi}{2}\left(e^{2\log_e 2} - e^0\right)$$

$$= \frac{\pi}{2}(4-1) = \frac{3\pi}{2}\text{ units}^3$$

Example 8

Find the volume generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x}}$ between $x = 4$ and $x = 9$.

Solution

$\text{Volume} = \pi\int_4^9 y^2\,dx \qquad \text{where } y = \frac{1}{\sqrt{x}}$

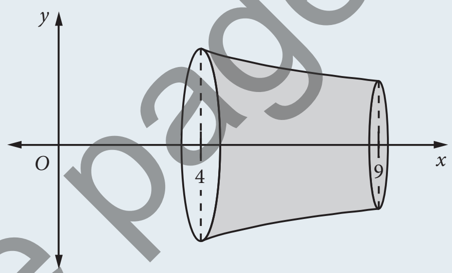
$$= \pi\int_4^9 \frac{1}{x}\,dx$$

$$= \pi\left[\log_e x\right]_4^9$$

$$= \pi\left(\log_e 9 - \log_e 4\right)$$

$$= \pi\log_e 2.25$$

$$\approx 2.548$$



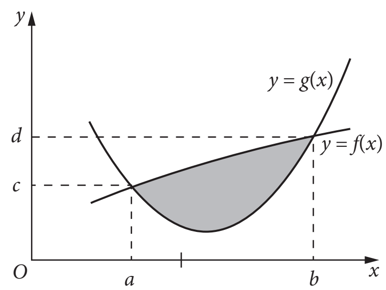
EXPLORING FURTHER

Solids of revolution

Use graphing software to graph and calculate the volume of solids of revolution.

Volume by rotating the region between two curves

When the region bounded by two curves $y = f(x)$ and $y = g(x)$ is rotated about the x-axis, the volume of the solid of revolution formed is given by $V = \pi\int_a^b\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$, where a and b are the abscissae of the points of intersection of the two curves, $a < b$ and $f(x) \geq g(x)$.

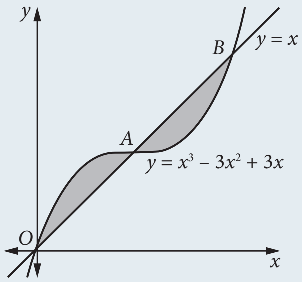


If the region is rotated about the y-axis, the equation of each curve must first be written as a function of y, i.e. $x = f-1(y)$ and $x = g-1(y)$, and the ordinates of the points of intersection used, namely c and d, as shown in the diagram.

The volume of the solid of revolution is given by $V = \pi\int_c^d\left(\left[f^{-1}(y)\right]^2 - \left[g^{-1}(y)\right]^2\right)dy$.

Example 9

The curve $y = x^3 - 3x^2 + 3x$ and the line $y = x$ intersect at (0, 0), A and B.



(a) Find the coordinates of A and B.

(b) Calculate the shaded area between the curves.

(c) The shaded region between the curves from O to A is rotated about the x-axis. Calculate the exact volume of the solid formed.

(d) The shaded region between the curves from A to B is rotated about the x-axis. Calculate the exact volume of the solid formed.

(e) Hence find the volume of the solid formed when the shaded region between the curves from O to B is rotated about the x-axis.

Solution

(a)

$$x^3 - 3x^2 + 3x = x$$
$$x^3 - 3x^2 + 2x = 0$$
$$x(x^2 - 3x + 2) = 0$$
$$x(x-1)(x-2) = 0$$
$$x = 0, 1, 2$$
$$y = 0, 1, 2$$

$A(1, 1)$ and $B(2, 2)$

(b)

$$\text{Area} = \int_0^1 \left(x^3 - 3x^2 + 3x - x\right)dx + \int_1^2 \left(x - \left(x^3 - 3x^2 + 3x\right)\right)dx$$
$$= \int_0^1 \left(x^3 - 3x^2 + 2x\right)dx + \int_1^2 \left(-x^3 + 3x^2 - 2x\right)dx$$
$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \left[-\frac{x^4}{4} + x^3 - x^2\right]_1^2$$
$$= \frac{1}{4} - 1 + 1 - 0 + \left(-4 + 8 - 4 - \left(-\frac{1}{4} + 1 - 1\right)\right)$$
$$= \frac{1}{2}\text{ unit}^2$$

(c)

$$\text{Volume from } O \text{ to } A = \pi\int_0^1 \left\{\left(x^3 - 3x^2 + 3x\right)^2 - x^2\right\}dx$$
$$= \pi\int_0^1 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right)dx$$
$$= \pi\left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_0^1$$
$$= \pi\left(\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3} - 0\right)$$
$$= \frac{13\pi}{42}\text{ units}^3$$

(d)

$$\text{Volume from } A \text{ to } B = \pi\int_1^2 \left\{x^2 - \left(x^3 - 3x^2 + 3x\right)^2\right\}dx$$
$$= -\pi\int_1^2 \left(x^6 - 6x^5 + 15x^4 - 18x^3 + 8x^2\right)dx$$
$$= -\pi\left[\frac{x^7}{7} - x^6 + 3x^5 - \frac{9x^4}{2} + \frac{8x^3}{3}\right]_1^2$$
$$= -\pi\left(\frac{128}{7} - 64 + 96 - 72 + \frac{64}{3} - \left[\frac{1}{7} - 1 + 3 - \frac{9}{2} + \frac{8}{3}\right]\right)$$
$$= \frac{29\pi}{42}\text{ units}^3$$

(e) Volume from O to $B = \dfrac{13\pi}{42} + \dfrac{29\pi}{42} = \pi\text{ units}^3$

EXERCISE 11.1 VOLUMES OF SOLIDS OF REVOLUTION

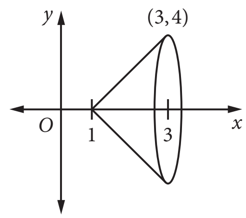
1 Find the volume of the solid of revolution formed by rotating about the x-axis the arc of the parabola $y = x^2$ between $x = 0$ and $x = 3$.

2 Find the volume of the solid of revolution formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = 4$.

3 A cone is formed by rotating about the x-axis a segment of the line $y = 3x$ between $x = 0$ and $x = 4$. The definite integral used to calculate the volume of this solid is:

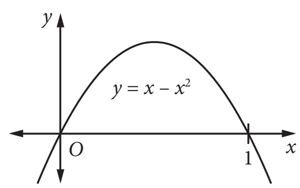
A $\int_0^4 9x^2\,dx$ **B** $\pi\int_0^4 3x^2\,dx$ **C** $\int_0^4 3x^2\,dx$ **D** $\pi\int_0^4 9x^2\,dx$

4 **(a)** Find the equation of the line passing through the points $(1, 0)$ and $(3, 4)$.
(b) A cone is formed by rotating about the x-axis the segment of the line joining the points $(1, 0)$ and $(3, 4)$. Calculate the volume of the cone.



5 The semicircle $y = \sqrt{9 - x^2}$ is rotated about the x-axis. Calculate the volume of the sphere generated.

6 The region bounded by the parabola $y = x - x^2$ and the x-axis is rotated about the x-axis. Find the volume of the solid formed.

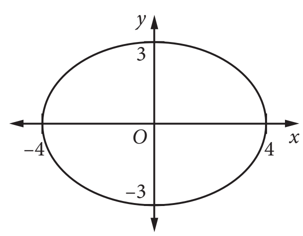


7 Find the volume of the solid formed when the region bounded by the parabola $y = 1 - x^2$ and the x-axis is rotated about: **(a)** the x-axis **(b)** the y-axis.

8 The region bounded by the parabola $y = (x - 2)^2$ and the coordinate axes is rotated about the x-axis. Find the volume of the solid generated.

9 Find the volume of the solid generated when the segment of the line joining the points $(0, 3)$ and $(6, 0)$ is rotated about: **(a)** the x-axis **(b)** the y-axis.

10 A rugby ball has a volume approximately the same as the volume generated by rotating the ellipse $9x^2 + 16y^2 = 144$ about the x-axis. Find its volume.

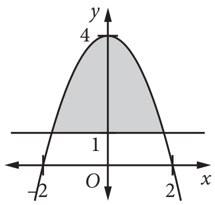


11 Find the volume of the solid formed when the region bounded by the parabola $y = 9 - x^2$ and the coordinate axes is rotated about: **(a)** the x-axis **(b)** the y-axis.

12 **(a)** Find the equation of the line through the points $(3, 0)$ and $(4, 10)$.
(b) A drinking glass has the shape of a truncated cone. The internal radii of the base and the top are 3 cm and 4 cm respectively and its depth is 10 cm. If the base of the glass sits on the x-axis, use integration to find its capacity.
(c) If the glass is filled with water to a depth of 5 cm, find the volume of water in the glass.

13 A hemispherical bowl of radius a units is filled with water to a depth of $\frac{a}{2}$ units. Use integration to find the volume of the water.

14 Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the line $y = 1$ is rotated about the y-axis.



15 A solid is formed by rotating about the y-axis the region bounded by the parabola $y = x^2 - 2$ and the x-axis. Indicate whether each statement below is a correct or incorrect step in calculating the volume of the solid formed.

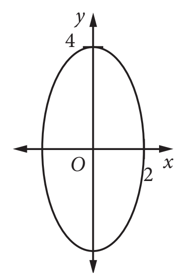
(a) $V = \pi\int_0^{\sqrt{2}}\left(x^2 - 2\right)^2 dx$ **(b)** $V = \pi\int_{-2}^{0}(y+2)\,dy$ **(c)** $V = \pi\left[\frac{y^2}{2} + 2y\right]_{-2}^{0}$ **(d)** $V = 2\pi$

16 Use integration to find the volume of the sphere generated when the circle $x^2 + y^2 = 16$ is rotated about the x-axis.

17 The area under the curve $y = 2x\sqrt{1 - x^2}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Using the trapezoidal rule with four subintervals, find an approximation for the volume of the solid correct to two decimal places.

18 Find the volume of the solid formed when the ellipse $4x^2 + y^2 = 16$ is rotated about:

(a) the x-axis **(b)** the y-axis.

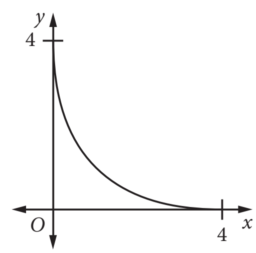


19 A region is bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes.

(a) Calculate the area of the region.

(b) Calculate the volume of the solid generated when the region is rotated about the x-axis.

(c) Calculate the volume of the solid generated when the region is rotated about the y-axis.



20 The region bounded by the curve $xy = 1$, the x-axis and the lines $x = 1$ and $x = a$, for $a > 1$, is rotated about the x-axis. Find V, the volume generated. Hence find $\lim_{a\to\infty} V$.

21 The area bounded by the parabola $y = 2x - x^2$, the y-axis and the line $y = 1$ is rotated about the x-axis. Find the volume generated.

22 Find the volume of the solid generated by rotating the region bounded by the parabola $y = 1 - x^2$ and the lines $x = 1$, $y = 1$ about: **(a)** the x-axis **(b)** the y-axis.

23 Find the volume of the cone formed by rotating the segment of the line $x + 2y = 4$ that is cut off by the axes about: **(a)** the x-axis **(b)** the y-axis.

24 Use the trapezoidal rule with five function values to estimate the volume of the solid formed by rotating the curve $y = \frac{1}{1+x^2}$ about the x-axis between $x = -2$ and $x = 2$.

25 The area under the curve $y = e^{-x}$ between $x = 0$ and $x = 1$ is rotated about the x-axis. Find the volume of the solid of revolution.

26 Find the volume generated when the curve $y = e^x$, $0.5 \le x \le 1.5$, is rotated about the x-axis.

27 Find the volume generated when the curve $y = e^{-0.5x}$, $-2 \le x \le 2$, is rotated about the x-axis.

28 Find the volume generated when the curve $y = e^x + e^{-x}$ between $x = -1$ and $x = 1$ is rotated about the x-axis.

29 **(a)** Find the area of the region bounded by the curve $y = e^{-x}$, the coordinate axes and the line $x = a$, $a > 0$.
(b) Find the limit of this area as $a \to \infty$.
(c) Find the volume of the solid generated by rotating the region in **(a)** about the x-axis and find the limit of this volume as $a \to \infty$.

30 Find the volume of the solid generated by rotating about the x-axis the region enclosed by the curve $y^2 = \frac{6}{x}$, the x-axis and the ordinates $x = 1$ and $x = 3$.

31 Find the volume of the solid generated by rotating about the x-axis the area beneath the curve $y = \frac{1}{\sqrt{x-2}}$ between $x = 6$ and $x = 11$.

32 **(a)** Given $a > 1$, sketch the curve $y = \log_e x$ for $1 \le x \le a$. Find the area enclosed by the curve and the lines $y = 0$ and $x = a$.
(b) The region enclosed by the curve $y = \log_e x$ and the lines $x = 0$, $y = \log_e a$ and $y = 0$ is rotated about the y-axis to form a solid of revolution. Find the volume of this solid.

33 Sketch the curve $y = \frac{1}{x^2}$ for values of x from $x = \frac{1}{2}$ to $x = 1$. This part of the curve is rotated about the y-axis to form a solid of revolution. Find its volume.

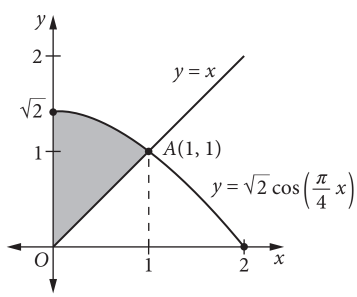
34 Sketch the curve $y = \frac{1}{\sqrt{4+x}}$ from $x = 0$ to $x = 5$. The region enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

35 The region enclosed by the curve $y = \frac{2}{\sqrt{x-7}}$ and the x-axis between $x = 8$ and $x = 10$ is rotated about the x-axis. Find the volume of the solid formed.

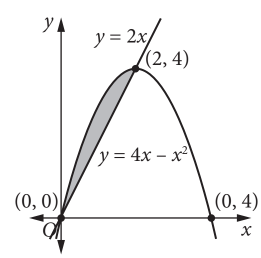
36 The region enclosed by the curve $y = \frac{\sqrt{x+1}}{x}$ and the x-axis between $x = 3$ and $x = 5$ is rotated about the x-axis. Find the volume of the solid formed.

37 **(a)** Sketch the region bounded by the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$.
(b) Calculate the area of the shaded region.
(c) The region bounded by the y-axis and the curves $y = 2(x^2 - 1)$ and $y = 1 - x^2$ for $x \ge 0$, is rotated about the y-axis. Calculate the volume of the solid of revolution generated.

38 The curve $y = \sqrt{2}\cos\left(\frac{\pi}{4}x\right)$ meets the line $y = x$ at the point $A(1, 1)$, as shown in the diagram.
(a) Find the exact value of the shaded area.
(b) The shaded area is rotated about the x-axis. Calculate the volume of the solid of revolution formed.
(c) The shaded area is rotated about the y-axis. Write the integral for this volume.
(d) By using a combination of exact integration and the trapezoidal rule, as appropriate, calculate the volume of the solid in **(c)**.



39 In the diagram on the right, the parabola $y = 4x - x^2$ and the line $y = 2x$ intersect at the points $(0, 0)$ and $(2, 4)$.
(a) Calculate the area of the region between the curves.
(b) The shaded region between the curves is rotated about the x-axis to form a solid of revolution. Calculate the exact volume of this solid.
(c) The shaded region between the curves is rotated about the y-axis to form a solid of revolution. Calculate the exact volume of this solid.



40 **(a)** Sketch the region bounded by the curve $y = 1 + \sqrt{x}$ and the lines $y = 1$ and $x = 4$.

(b) Calculate the area of this region.

(c) This region is rotated around the x-axis to form a solid. Calculate the volume of this solid.

(d) Calculate the volume of the solid formed if this region is rotated about the y-axis.

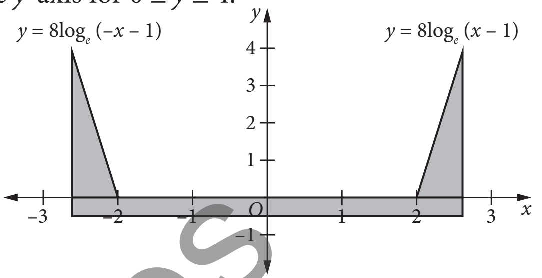
41 **(a)** On the same diagram sketch the graphs of $x^2 + y^2 = 1$ for $-1 \le x \le 0$ and $\frac{x^2}{4} + y^2 = 1$ for $0 \le x \le 2$.

(b) An egg is modelled by rotating about the x-axis the curves $x^2 + y^2 = 1$ for $-1 \le x \le 0$ and $\frac{x^2}{4} + y^2 = 1$ for $0 \le x \le 2$ to form a solid of revolution. Find the exact value of the volume of the egg.

42 A bowl is formed by rotating the curve $y = 8\log_e(x - 1)$ about the y-axis for $0 \le y \le 4$.

(a) Calculate the volume of the bowl (capacity), giving your answer correct to one decimal place.

(b) This bowl is to be moulded out of plastic with vertical sides and a solid base 0.5 units thick. The cross-section of the bowl is shown in the diagram, right. Calculate the volume of plastic used to make the bowl.



11.2 INDEFINITE INTEGRALS AND SUBSTITUTION

Some integrals can only be solved using particular substitutions for the variables. In this Mathematics Extension 1 course, any substitutions needed to find an integral are given.

Integration using a substitution can be considered as the converse of the method of differentiating a composite function—it's like using the chain rule backwards.

The aim of a substitution is to transform an integral into one that involves a standard result, e.g. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$. Variable substitution works as follows:

$$\text{Let} \quad y = \int f(u)\,du \quad \text{where } u = g(x)$$

$$\therefore \quad \frac{dy}{du} = f(u)$$

$$\text{But} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= f(u) \times \frac{du}{dx}$$

$$\therefore \quad y = \int f(u) \times \frac{du}{dx}\,dx$$

$$\int f(u) \times \frac{du}{dx}\,dx = \int f(u)\,du$$

This 'backwards' form of the chain rule is convenient when the substitution of $u = g(x)$ allows a function to be expressed as the product of $\frac{du}{dx}$ and a function of u. For example:

- If $f(x) = 2x^2(x^3 - 1)^4$ then you can substitute $u = x^3 - 1$. As $\frac{du}{dx} = 3x^2$, you can write $2x^2$ as $\frac{2}{3} \times 3x^2$, so that $f(x)$ is written: $f(x) = \frac{2}{3}(x^3 - 1)^4 \times (3x^2)$

  $= \frac{2}{3}u^4 \frac{du}{dx}$ where $u = x^3 - 1$

- If $f(x) = x\sqrt{1 + x^2}$, you can see that $2x$ is the derivative of $1 + x^2$, so if you make the substitution $u = 1 + x^2$ and write x as $\frac{1}{2}(2x)$, then $f(x) = \frac{1}{2}u^{\frac{1}{2}}\frac{du}{dx}$ where $u = 1 + x^2$ and $\frac{du}{dx} = 2x$.

- If $f(x)=\dfrac{x+1}{(x^2+2x)^3}$, you can see that $2x+2$ is the derivative of x^2+2x, so if you make the substitution $u=x^2+2x$ and write $x+1$ as $\frac{1}{2}(2x+2)$, then $f(x)=\frac{1}{2}u^{-3}\frac{du}{dx}$.
- If $f(x)=x\sqrt{1-x}$, then you can make the substitution $u=1-x$. As $x=1-u$, $\frac{du}{dx}=-1$, so:

$$\begin{aligned}
f(x)&=-x\sqrt{1-x}\times(-1)\\
&=-(1-u)u^{\frac{1}{2}}\times\frac{du}{dx}\\
&=-\left(u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)\times\frac{du}{dx}\\
&=\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)\times\frac{du}{dx}\\
&=f(u)\frac{du}{dx}
\end{aligned}$$

Example 10

Find: (a) $\int 3x^2\left(x^3-1\right)^4 dx$ using the substitution $u=x^3-1$

(b) $\int x\sqrt{1+x^2}\,dx$ using the substitution $u=1+x^2$

(c) $\int \dfrac{x+1}{\left(x^2+2x\right)^3}dx$ using the substitution $u=x^2+2x$.

Solution

(a) $u=x^3-1,\ \frac{du}{dx}=3x^2$

$$\begin{aligned}
\int 3x^2\left(x^3-1\right)^4 dx&=\int u^4\times\frac{du}{dx}dx\\
&=\int u^4\,du\\
&=\frac{1}{5}u^5+C\\
&=\frac{1}{5}\left(x^3-1\right)^5+C
\end{aligned}$$

(b) $u=1+x^2,\ \frac{du}{dx}=2x$

$$\begin{aligned}
\int x\sqrt{1+x^2}\,dx&=\frac{1}{2}\int 2x\sqrt{1+x^2}\,dx\\
&=\frac{1}{2}\int u^{\frac{1}{2}}\times\frac{du}{dx}dx\\
&=\frac{1}{2}\int u^{\frac{1}{2}}\,du\\
&=\frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}+C\\
&=\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+C
\end{aligned}$$

(c) $u=x^2+2x,\ \frac{du}{dx}=2x+2$

$$\begin{aligned}
\int\frac{x+1}{\left(x^2+2x\right)^3}dx&=\frac{1}{2}\int(2x+2)\left(x^2+2x\right)^{-3}dx\\
&=\frac{1}{2}\int u^{-3}\times\frac{du}{dx}dx\\
&=\frac{1}{2}\int u^{-3}\,du\\
&=\frac{1}{2}\times\left(\frac{1}{-2}\right)u^{-2}+C\\
&=\frac{-1}{4\left(x^2+2x\right)^2}+C
\end{aligned}$$

A quick way to check your answer is to differentiate it to see that it gives the integrand.

Example 11

Find: (a) $\int x\sqrt{1-x}\,dx$ using the substitution $u = 1 - x$ (b) $\int \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(c) $\int (3x-5)^4\,dx$ using the substitution $u = 3x - 5$.

Solution

(a) $u = 1 - x,\ \frac{du}{dx} = -1,\ x = 1 - u$

$$\int x\sqrt{1-x}\,dx = \int (1-u)u^{\frac{1}{2}} \times \frac{du}{dx}\,dx$$
$$= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)(-1)\,dx$$
$$= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du \qquad \text{(note that } du = (-1)\,dx\text{)}$$
$$= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C$$
$$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$$

(b) $u = 1 + t,\ \frac{du}{dt} = 1,\ t = u - 1$

$$\int \frac{t}{\sqrt{1+t}}\,dt = \int \frac{u-1}{\sqrt{u}} \times \frac{du}{dt} \times dt$$
$$= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)(1)\,dt$$
$$= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \qquad \text{(note that } du = (1)\,dt\text{)}$$
$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$$
$$= \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C$$

(c) $u = 3x - 5,\ \frac{du}{dx} = 3$

$$\int (3x-5)^4\,dx = \frac{1}{3}\int 3(3x-5)^4\,dx$$
$$= \frac{1}{3}\int u^4 \times \frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int u^4\,du$$
$$= \frac{1}{3} \times \frac{1}{5}u^5 + C$$
$$= \frac{1}{15}(3x-5)^5 + C$$

EXERCISE 11.2 INDEFINITE INTEGRALS AND SUBSTITUTION

1 Find: (a) $\int 2x\left(x^2-1\right)^4\,dx$ using the substitution $u = x^2 - 1$

(b) $\int 3x^2\left(x^3+4\right)^3\,dx$ using the substitution $u = x^3 + 4$

(c) $\int x^2\sqrt{x^3+1}\,dx$ using the substitution $u = x^3 + 1$.

2 Find: (a) $\int (2t+1)^3\,dt$ using the substitution $u = 2t + 1$

(b) $\int \frac{2x}{\sqrt{x^2-4}}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int (2x+1)\left(x^2+x+2\right)^5\,dx$ using the substitution $u = x^2 + x + 2$.

3 Using $u = 2x + 3$, $\int \frac{dx}{(2x+3)^3} = \ldots$

A $4(2x+3)^4 + C$ **B** $\frac{1}{4(2x+3)^4} + C$ **C** $-4(2x+3)^2 + C$ **D** $\frac{-1}{4(2x+3)^2} + C$

4 Find: **(a)** $\int (3-2x)^6\,dx$ using the substitution $u = 3 - 2x$

(b) $\int \frac{3x+1}{\left(3x^2+2x+5\right)^2}\,dx$ using the substitution $u = 3x^2 + 2x + 5$

(c) $\int \left(x^2-2x\right)\left(x^3-3x^2+1\right)^4\,dx$ using the substitution $u = x^3 - 3x^2 + 1$.

5 Find: **(a)** $\int 3x^2\left(x^3+1\right)^4\,dx$ using the substitution $u = x^3 + 1$

(b) $\int \frac{t}{\sqrt{1-t^2}}\,dt$ using the substitution $u = 1 - t^2$

(c) $\int (3x-5)^{\frac{2}{3}}\,dx$ using the substitution $u = 3x - 5$.

6 Find: **(a)** $\int 2t\sqrt{1-t^2}\,dt$ using the substitution $u = 1 - t^2$

(b) $\int x\sqrt{a^2-x^2}\,dx$ using the substitution $u = a^2 - x^2$

(c) $\int z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2 + 1$.

7 Find: **(a)** $\int y\sqrt{y+1}\,dy$ using the substitution $u = y + 1$

(b) $\int \frac{x}{(x-1)^3}\,dx$ using the substitution $u = x - 1$

(c) $\int \frac{x}{\sqrt{2x-1}}\,dx$ using the substitution $u = 2x - 1$.

8 **(a)** Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = 1 + x^3$.

(b) Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = x^3$.

(c) Why is the substitution in **(a)** easier to use than the substitution in **(b)**?

9 $\frac{dy}{dx} = x\sqrt{x^2-4}$ and $y = 2$ at $x = \sqrt{5}$. Use the substitution $u = x^2 - 4$ to find y in terms of x.

10 If $f'(x) = \frac{3x}{\sqrt{x^2+1}}$ for all x and $f(0) = 2$, use the substitution $u = x^2 + 1$ to find $f(x)$.

11 $\frac{dx}{dt} = \frac{t-1}{\sqrt{t^2-2t+4}}$ and $x = 10$ when $t = 0$. Use the substitution $u = t^2 - 2t + 4$ to find x in terms of t.

12 Given that $\frac{dr}{d\theta} = \frac{3}{(1-r)^4}$ and $r(0) = 0$, use the substitution $u = 1 - r$ to find r in terms of θ.

13 At any point where $x > \frac{1}{2}$, the gradient of a curve is given by $\frac{dy}{dx} = \sqrt{2x-1}$. If the point $(2.5, 9)$ is on the curve, use the substitution $u = 2x - 1$ to find the equation of the curve.

14 Given that $\frac{dt}{dx} = \frac{1}{2(4-x)^2}$ and $x = 0$ when $t = 0$, use the substitution $u = 4 - x$ to find x in terms of t.

11.3 DEFINITE INTEGRALS AND SUBSTITUTION

When using a substitution to evaluate a definite integral you must take care with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Example 12

Evaluate: (a) $\int_1^2 2x\sqrt{x^2-1}\,dx$ using the substitution $u = x^2 - 1$

(b) $\int_{-5}^3 x\sqrt{4-x}\,dx$ using the substitution $u = 4 - x$

(c) $\int_0^1 x^2\left(x^3+1\right)^4 dx$ using the substitution $u = x^3 + 1$.

Solution

(a) $u = x^2 - 1,\ \dfrac{du}{dx} = 2x$

Limits: for $x = 1$, $u = 1^2 - 1 = 0$
for $x = 2$, $u = 2^2 - 1 = 3$

$$\int_1^2 2x\sqrt{x^2-1}\,dx = \int_0^3 \sqrt{u}\times\frac{du}{dx}dx$$
$$= \int_0^3 u^{\frac{1}{2}}\,du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3$$
$$= \frac{2}{3}\left(3^{\frac{3}{2}} - 0\right) = 2\sqrt{3}$$

(c) $u = x^3 + 1,\ \dfrac{du}{dx} = 3x^2$

Limits: for $x = 0$, $u = 1$
for $x = 1$, $u = 1^3 + 1 = 2$

$$\int_0^1 x^2\left(x^3+1\right)^4 dx = \frac{1}{3}\int_1^2 u^4\times\frac{du}{dx}dx$$
$$= \frac{1}{3}\int_1^2 u^4\,du$$
$$= \frac{1}{3}\left[\frac{1}{5}u^5\right]_1^2$$
$$= \frac{1}{15}\left[u^5\right]_1^2$$
$$= \frac{1}{15}(32-1) = \frac{31}{15}$$

(b) $u = 4 - x,\ \dfrac{du}{dx} = -1$

$x = 4 - u$, so $x\sqrt{4-x} = (4-u)\sqrt{u}$

or $-x\sqrt{4-x} = (u-4)\sqrt{u}$

Limits: for $x = -5$, $u = 4 + 5 = 9$
for $x = 3$, $u = 4 - 3 = 1$

$$\int_{-5}^3 x\sqrt{4-x}\,dx = \int_9^1 (u-4)\sqrt{u}\times\frac{du}{dx}dx$$
$$= \int_9^1\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} - 4\times\frac{2}{3}u^{\frac{3}{2}}\right]_9^1$$
$$= \left(\frac{2}{5}-\frac{8}{3}\right) - \left(\frac{2}{5}\times 9^{\frac{5}{2}} - \frac{8}{3}\times 9^{\frac{3}{2}}\right)$$
$$= \frac{2}{5} - \frac{8}{3} - \frac{2}{5}\times 3^5 + \frac{8}{3}\times 3^3$$
$$= -\frac{412}{15}$$

Useful result

$\int_a^b f(x)dx = -\int_b^a f(x)dx$ If you reverse the limits of integration, you change the sign of the integral.

In Example **11**(b) the integral $\int_9^1\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$ could have been written as $-\int_1^9\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$.

Check that this integral gives the same answer.

EXERCISE 11.3 DEFINITE INTEGRALS AND SUBSTITUTION

1 Evaluate: (a) $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $u = 1 - x^2$

(b) $\int_{-1}^2 x\sqrt{2-x}\,dx$ using the substitution $u = 2 - x$

(c) $\int_0^2 \dfrac{2x}{\sqrt{x^2+1}}dx$ using the substitution $u = x^2 + 1$.