NEW SENIOR

Mathematics

EXTENSION 2 YEAR 12 | 4TH EDITION

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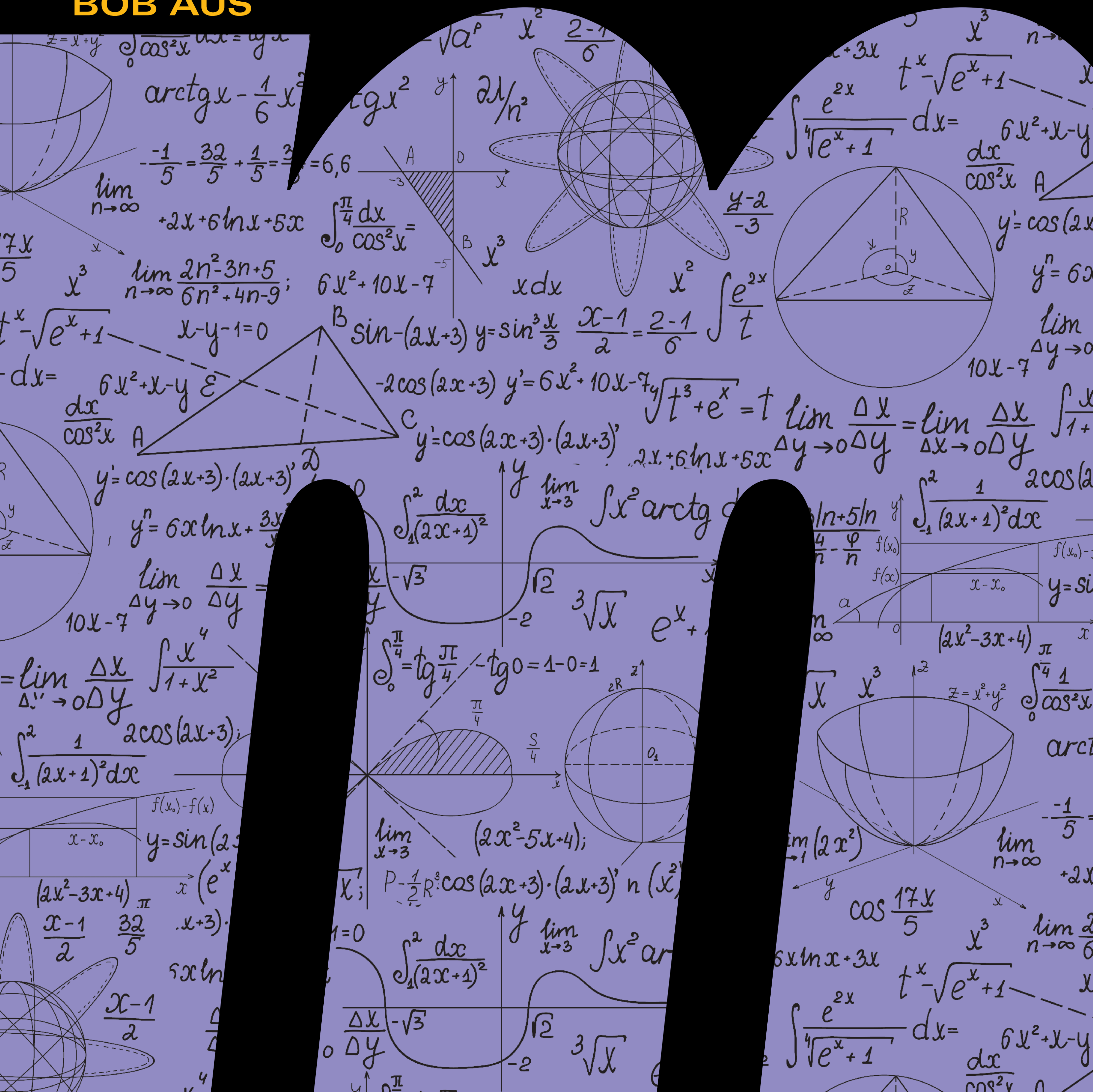
NSW STAGE 6



CHAPTER 6
Mechanics

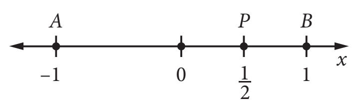
6.1 VELOCITY AND ACCELERATION AS FUNCTIONS OF *x*

The expressions for displacement $x(t)$, velocity $v(t)=\dot{x}(t)$ and acceleration $a(t)=\ddot{x}(t)=\dot{v}(t)$ all clearly define functions of time t. This makes sense because an ordinary particle cannot be in two different places at the same time: for each value of t there corresponds only one value of $x(t)$ that satisfies the function. Similarly, $v(t)$ and $a(t)$ define functions because a particle cannot be moving in two different ways (that is, cannot have more than one velocity and one acceleration) at any given time.

However, sometimes it is still useful to express acceleration in terms of the displacement x or in terms of the velocity v, rather than in terms of time t. In this case you must be careful, because equations relating a, v and x will not necessarily define functions.

Consider a particle that moves so that its displacement at time $t \geq 0$ is given by $x(t)=\sin t$. This means that the particle is moving back and forth between the limits $x=\pm 1$. (This is an example of simple harmonic motion, an important kind of motion that is explored in more detail later in this chapter.)

The particle is at $x=\frac{1}{2}$ whenever $\sin t=\frac{1}{2}$, i.e. when $t=\frac{\pi}{6}, \pi-\frac{\pi}{6}, 2\pi+\frac{\pi}{6}, \ldots$ so that $x(t)$ defines a many-to-one function of time. Similarly, $v(t)=\dot{x}(t)=\cos t$ defines a many-to-one function of time.



Note that $v^2=\cos^2 t=1-\sin^2 t=1-x^2$. Hence the equation $v^2=1-x^2$ expresses v in terms of x. However, this equation is not a function of x, because (for example) at $x=\frac{1}{2}$ you may have $v=\pm\frac{\sqrt{3}}{2}$. The point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ corresponds to $t=\frac{\pi}{6}$, while $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ corresponds to $t=\frac{5\pi}{6}$. There is an infinite number of values of t for each value of x. As the particle moves to and fro between A and B, its velocity has two possible values at any point P in between, depending on whether the particle is moving towards A or towards B at the time. At the extremes A and B, the velocity is zero, so at these points you can say the particle is instantaneously at rest.

Remember that differential calculus needs to be applied to functions. In this context, $\frac{dv}{dx}$ can only have a useful meaning if $v=v(x)$ defines a function. Hence if you want to differentiate the equation $v^2=1-x^2$ you must restrict v to be either positive or negative. This means that in the equation $v^2=1-x^2$ you consider v to apply only to velocities as the particle moves from A to B, i.e. $v>0$, or only from B to A, i.e. $v<0$.

If $v(x)$ specifies a velocity function of x according to one of these restrictions, then you can use the chain rule of differentiation to calculate acceleration:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx}\times\frac{dx}{dt} \\ &= v\frac{dv}{dx} \quad \text{as } v=\frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\times\frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a=f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a=g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

- Given $a = h(v)$, use $\begin{cases} \dfrac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\dfrac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

For velocity:

- Given $v = f(t)$, use $v = \dfrac{dx}{dt}$
- Given $v = g(x)$, use $v = \dfrac{dx}{dt}$ and consider using $\dfrac{dt}{dx}$

It is customary to write derivatives with respect to time using dots above the dependent variable, e.g. $\dot{x} = \dfrac{dx}{dt}, \ddot{x} = \dfrac{d^2x}{dt^2}, \dot{v} = \dfrac{dv}{dt}$.

Example 1

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If the acceleration is $3 - 2x$, find v in terms of x given that $v = 2$ when $x = 1$.

Solution

$\ddot{x} = 3 - 2x$ gives the acceleration as a function of x and the initial conditions are v and x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3 - 2x$

Integrate with respect to x: $\quad \dfrac{1}{2}v^2 = \int (3 - 2x)\,dx$

$$\frac{1}{2}v^2 = 3x - x^2 + C$$

At $x = 1$, $v = 2$: $\quad 2 = 3 - 1 + C$

$$C = 0$$

$$\therefore \quad \frac{1}{2}v^2 = 3x - x^2$$

$$v^2 = 6x - 2x^2$$

$$v = \pm\sqrt{6x - 2x^2}$$

The conditions include $x = 1$, $v = 2$, which means that v is positive when $x = 1$.
To satisfy the initial conditions you must take the positive square root.

$$\therefore \quad v = \sqrt{6x - 2x^2}$$

It is worth noting the domain of the velocity function. For v to exist, $6x - 2x^2 \geq 0$, so $0 \leq x \leq 3$: this means that the motion exists only for $0 \leq x \leq 3$.

Example 2

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = 3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

Solution

$\ddot{x} = 3x^2$ gives acceleration as a function of x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3x^2$

Integrate with respect to x: $\frac{1}{2}v^2 = \int 3x^2 dx$

$$\frac{1}{2}v^2 = x^3 + C_1$$

At $x = 1, v = -\sqrt{2}$: $1 = 1 + C_1$

$$C_1 = 0$$

$\therefore$ $\frac{1}{2}v^2 = x^3$

$$v^2 = 2x^3$$

$$v = \pm\sqrt{2x^3}$$

The conditions include $x = 1, v = -\sqrt{2}$, which means that v is negative when $x = 1$.
To satisfy the initial conditions you must take the negative square root.

$\therefore$ $v = -\sqrt{2}x^{\frac{3}{2}}$

Now $v = \frac{dx}{dt}$: $\frac{dx}{dt} = -\sqrt{2}x^{\frac{3}{2}}$

Reciprocal of both sides: $\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$

Integrate with respect to t: $t = -\frac{1}{\sqrt{2}}\int x^{-\frac{3}{2}}dx$

$$t = -\frac{1}{\sqrt{2}} \times \left(-\frac{2}{1}\right)x^{-\frac{1}{2}} + C_2$$

$$t = \sqrt{2}x^{-\frac{1}{2}} + C_2$$

At $x = 1, t = 0$: $0 = \sqrt{2} + C_2$

$$C_2 = -\sqrt{2}$$

$\therefore$ $t = \sqrt{2}x^{-\frac{1}{2}} - \sqrt{2}$

Find x in terms of t: $\sqrt{2}x^{-\frac{1}{2}} = t + \sqrt{2}$

$$x^{-\frac{1}{2}} = \frac{t + \sqrt{2}}{\sqrt{2}}$$

$$x^{\frac{1}{2}} = \frac{\sqrt{2}}{t + \sqrt{2}}$$

$$x = \frac{2}{\left(t + \sqrt{2}\right)^2}$$

Example 3

The velocity of a particle is $v = 3x + 7\text{ m s}^{-1}$.

(a) Find an expression for the acceleration.

(b) If the initial displacement is 1 m to the right of the origin, find the displacement as a function of time.

Solution

(a) $v = 3x + 7$

$$\frac{dv}{dx} = 3$$

Use $\ddot{x} = v\frac{dv}{dx}$: $\ddot{x} = (3x+7) \times 3$

$$= 3(3x+7)\ \text{m s}^{-2}$$

(b)

$$\frac{dx}{dt} = 3x + 7$$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{3x+7}$

Integrate with respect to x: $t = \int \frac{dx}{3x+7}$

$$t = \frac{1}{3}\log_e(3x+7) + C$$

At $t = 0$, $x = 1$: $C = -\frac{1}{3}\log_e 10$

$\therefore$ $t = \frac{1}{3}\log_e(3x+7) - \frac{1}{3}\log_e 10$

$$t = \frac{1}{3}\log_e\left(\frac{3x+7}{10}\right)$$

$$3t = \log_e\left(\frac{3x+7}{10}\right)$$

$$\left(\frac{3x+7}{10}\right) = e^{3t}$$

$$3x + 7 = 10e^{3t}$$

$$x = \frac{1}{3}\left(10e^{3t} - 7\right)$$

Example 4

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = -e^{-\frac{x}{2}}$ and $v = 2$, $x = 0$ when $t = 0$, find x as a function of t.

Solution

Use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-\frac{x}{2}}$

Integrate with respect to x: $\frac{1}{2}v^2 = -\int e^{-\frac{x}{2}}dx$

$$\frac{1}{2}v^2 = 2e^{-\frac{x}{2}} + C_1$$

At $x = 0$, $v = 2$: $2 = 2 + C_1$

$$C_1 = 0$$

$\therefore$ $\frac{1}{2}v^2 = 2e^{-\frac{x}{2}}$

$$v^2 = 4e^{-\frac{x}{2}}$$

$$v = \pm 2e^{-\frac{x}{4}}$$

The conditions include $x = 0$, $v = 2$, which means that v is positive when $x = 0$; also $e^{-\frac{x}{4}} > 0$ for all x. To satisfy the initial conditions you must take the positive square root.

$\therefore$ $v = 2e^{-\frac{x}{4}}$

Now $v = \frac{dx}{dt}$: $\frac{dx}{dt} = 2e^{-\frac{x}{4}}$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{2}e^{\frac{x}{4}}$

Integrate with respect to x: $t = \frac{1}{2}\int e^{\frac{x}{4}} dx$

$$t = \frac{1}{2} \times 4e^{\frac{x}{4}} + C_2$$

$$t = 2e^{\frac{x}{4}} + C_2$$

At $t = 0$, $x = 0$: $0 = 2 + C_2$

$$C_2 = -2$$

$\therefore$ $t = 2e^{\frac{x}{4}} - 2$

Find x in terms of t: $e^{\frac{x}{4}} = \frac{t+2}{2}$

$$\frac{x}{4} = \log_e\left(\frac{t+2}{2}\right)$$

$$x = 4\log_e\left(\frac{t+2}{2}\right)$$

Example 5

A particle is initially at the origin and is travelling at a speed of 3 m s^{-1} to the left. Its acceleration is given by $a = 6x + 5$, where x is the particle's displacement from the origin at time t. Find its velocity when it is 4 metres to the left of the origin.

Solution

$t = 0$, $x = 0$, $v = -3$, $a = 6x + 5$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x + 5$$

$$\frac{1}{2}v^2 = \int (6x+5)\,dx$$

$$\frac{1}{2}v^2 = 3x^2 + 5x + C$$

$$x = 0, v = -3: \frac{9}{2} = C$$

$$v^2 = 6x^2 + 10x + 9$$

$$v = \pm\sqrt{6x^2 + 10x + 9}$$

$$x = 0, v = -3: v = -\sqrt{6x^2 + 10x + 9}$$

$$x = -4: v = -\sqrt{96 - 40 + 9} = -\sqrt{65}\text{ m s}^{-1}$$

Example 6

If $\ddot{x}=\frac{1}{2}\left(v^2+1\right)$, find v in terms of x given that $v=1$ when $x=0$.

Solution

$\ddot{x}=\frac{1}{2}\left(v^2+1\right)$ gives acceleration as a function of v, so to find v as a function of x use $\ddot{x}=v\frac{dv}{dx}$.

$$v\frac{dv}{dx}=\frac{1}{2}\left(v^2+1\right)$$

$$\frac{dv}{dx}=\frac{v^2+1}{2v}$$

$$\frac{dx}{dv}=\frac{2v}{v^2+1}$$

$$x=\int\frac{2v}{v^2+1}\,dv$$

$$x=\ln\left|v^2+1\right|+C$$

$v=1, x=0$: $\quad 0=\ln 2+C$

so $\quad C=-\ln 2$

$v^2+1>0$: $\quad x=\ln\left(v^2+1\right)-\ln 2$

$$x=\ln\left(\frac{v^2+1}{2}\right)$$

$$e^x=\frac{v^2+1}{2}$$

$$v^2=2e^x-1$$

so $\quad v=\pm\sqrt{2e^x-1}$

Given $v=1, x=0$ so take the positive square root giving $v=\sqrt{2e^x-1}$.

Example 7

A particle moves in a straight line so that at time t, its displacement from a fixed origin is x and its velocity is v. If the acceleration is:

(a) $-3v^2$, find v in terms of t, given that $v=1$ when $t=0$;

(b) $4-v^2$, find t in terms of v, given that $v(0)=0$ and hence find v as a function of t.

Solution

(a) $\ddot{x}=-3v^2$, use: $\frac{dv}{dt}=-3v^2$

Reciprocals: $\frac{dt}{dv}=-\frac{1}{3v^2}, v\neq 0$

$$t=-\frac{1}{3}\int v^{-2}\,dv$$

$$=\frac{1}{3}v^{-1}+C$$

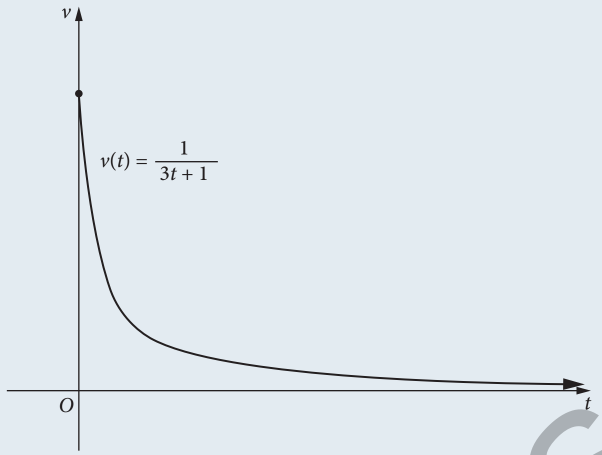
$v(0) = 1$: $\quad C = -\frac{1}{3}$

$$t = \frac{1}{3}\left(\frac{1}{v} - 1\right)$$

$$3t + 1 = \frac{1}{v}$$

$$v = \frac{1}{3t+1}, t \geq 0$$

Note that $v(t) \to 0$ as $t \to \infty$.
Since $v(0) = 1$, then $v \in (0, 1)$.



(b) $\ddot{x} = 4 - v^2$, use: $\quad \frac{dv}{dt} = 4 - v^2$

Reciprocals: $\quad \frac{dt}{dv} = \frac{1}{4 - v^2}$

Partial fractions: $\quad \frac{1}{4 - v^2} = \frac{a}{2+v} + \frac{b}{2-v}$

$$1 = a(2 - v) + b(2 + v)$$

$v = 2$: $1 = 4b \Rightarrow \quad b = \frac{1}{4}$

$v = -2$: $1 = 4a \Rightarrow \quad a = \frac{1}{4}$

Or by inspection: $\quad \frac{1}{4 - v^2} = \frac{1}{4}\left(\frac{1}{2+v} + \frac{1}{2-v}\right)$

Hence $\quad t = \frac{1}{4}\int\left(\frac{1}{2+v} + \frac{1}{2-v}\right)dv$

$t = \frac{1}{4}(\ln(2+v) - \ln(2-v)) + C$ where both $2 + v > 0$ and $2 - v > 0$.

Hence $\quad -2 < v < 2$

So $\quad t = \frac{1}{4}\ln\left(\frac{2+v}{2-v}\right) + C$ where $-2 < v < 2$.

$v(0) = 0$: $\quad C = 0$

$$t = \frac{1}{4}\ln\left(\frac{2+v}{2-v}\right)$$

Since $t \geq 0$ then $\ln\left(\frac{2+v}{2-v}\right) \geq 0$ so $\left(\frac{2+v}{2-v}\right) \geq 1$ for the motion to occur and $0 \leq v < 2$

$$4t = \ln\left(\frac{2+v}{2-v}\right)$$

$$e^{4t} = \frac{2+v}{2-v}$$

$$2e^{4t} - ve^{4t} = 2 + v$$

$$v\left(1 + e^{4t}\right) = 2\left(e^{4t} - 1\right)$$

$$v = \frac{2\left(e^{4t} - 1\right)}{e^{4t} + 1} \text{ for } t \geq 0.$$

This may be written as $v = \frac{2\left(1 - e^{-4t}\right)}{1 + e^{-4t}}$ and thus as $t \to \infty$ then $v \to 2$.

EXERCISE 6.1 VELOCITY AND ACCELERATION AS FUNCTIONS OF x

1 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is given by $\ddot{x} = 4 + x$ and $v = 1$ when $x = 0$, find v when $x = 1$.

2 The acceleration of a particle moving in a straight line is given by $\dfrac{d^2x}{dt^2} = 7 - 2x$ and the particle starts from rest at the point where $x = 0$. Find the speed in terms of x. What values can x take?

3 At time t, the displacement of a particle moving in a straight line is x. If the acceleration is given by $\dfrac{d^2x}{dt^2} = 3 - 4x$ and the particle starts from rest at $x = 1$, find its velocity at any position. At what other point, if any, does the particle come to rest?

4 The acceleration of a particle moving in a straight line is given by $\ddot{x} = kx$ where $k = 1$. If $\dot{x} = 10\,\text{m}\,\text{s}^{-1}$ when $x = 6\,\text{m}$, the velocity of the particle when it is 15 m from the origin is:

A $-17\,\text{m}\,\text{s}^{-1}$ **B** $\sqrt{79}\,\text{m}\,\text{s}^{-1}$ **C** $17\,\text{m}\,\text{s}^{-1}$ **D** $23\,\text{m}\,\text{s}^{-1}$

5 A particle leaves O with velocity $2\,\text{m}\,\text{s}^{-1}$. Its acceleration is $-\dfrac{1}{6}\sqrt[3]{x}\,\text{m}\,\text{s}^{-2}$ when its displacement from O is x metres. Find its displacement when it first comes to rest.

6 A particle moves in a straight line and its acceleration at any time t is $\cos x$. If $v = 0$ and $x = 0$ when $t = 0$, express v in terms of x.

7 A particle moves in a straight line and its acceleration at any time is given by $\dfrac{d^2x}{dt^2} = \sin^2 x$. Find $\dfrac{dx}{dt}$ given that $\dfrac{dx}{dt} = 1$ when $x = 0$.

8 If $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 2x - 3x^2$ and $v = 2$ when $x = 0$, find v in terms of x.

9 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\ddot{x} = x - 3$ and $\dot{x} = -2$ when $x = 1$, find $\dot{x}$ when $x = 0$.

10 The velocity of a particle is given by $v = 4 + x^2\,\text{m}\,\text{s}^{-1}$.

(a) Find the acceleration as a function of x.

(b) If initially $x = -2\,\text{m}$, what is the displacement after $\dfrac{\pi}{4}$ seconds?

11 If $\dfrac{dx}{dt} = \dfrac{1}{x+4}$ and $x = 0$ when $t = 0$, find t when $x = 2$.

12 If $\dfrac{dx}{dt} = x + 4$ and $x = -3$ when $t = 0$, express x in terms of t.

13 The velocity of a particle at any time t is $2(4 - x)^2$. Find its position $x(t)$ at any time t given $x(0) = 0$.

14 If $\dfrac{dx}{dt} = (3 - x)^2$ and $x = 2$ when $t = 0$, find: **(a)** x as a function of t **(b)** $\dfrac{d^2x}{dt^2}$ as a function of x.

15 A particle moves in a straight line and, at time t, the displacement from a fixed origin is x. If $\ddot{x} = 3 + v$ and $v = 0$ when $t = 0$, find v in terms of t.

16 A particle moves in a straight line and, at time t, the displacement from a fixed origin is x. If $\ddot{x} = 3 - v$ and $v = 0$ when $t = 0$, find the time at which $v = 2$.

17 If $\ddot{x} = \dfrac{1}{2}\left(v^2 + 1\right)$, find v in terms of x given that $v = 2$ when $x = 0$.

18 A particle moves in a straight line so that, at time t, its displacement from a fixed origin is x and its velocity is v. If the acceleration is:

(a) $-6v^2$, find v in terms of t, given that $v = 5$ when $t = 0$

(b) $9 - v^2$, find t in terms of v, given that $v(0) = 0$ and hence find v as a function of t.

19 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\dot{x} = x + 3$:

(a) express $\ddot{x}$ in terms of x **(b)** find x when $t = 1$, given that $x = -2$ when $t = 0$.

20 The acceleration of a body moving under gravitational attraction towards a planet varies inversely as the square of its distance from the centre of the planet. This can be written as $\frac{d^2x}{dt^2} = -\frac{k}{x^2}$ where x is the distance from the centre of the planet and k is a constant. If the body starts from rest at a distance a from the centre of the planet, show that its speed at x (before it hits the planet) is given by $\frac{dx}{dt} = \sqrt{\frac{2k(a-x)}{ax}}$.

21 A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x} = -e^{-2x}$. It is initially at the origin and travelling with a velocity of 1 metre per second.

(a) Show that $\dot{x} = e^{-x}$. **(b)** Hence show that $x = \log_e(t+1)$.

22 A particle is moving so that $\ddot{x} = 32x^3 + 48x^2 + 16x$. Initially $x = -2$ and the velocity v is -8.

(a) Show that $v^2 = 16x^2(1+x)^2$. **(b)** Hence, or otherwise, show that $-4t = \int \frac{1}{x(1+x)}\,dx$.

(c) It can be shown that for some constant C, $\log_e\left(1+\frac{1}{x}\right) = 4t + C$. Using this equation and the initial conditions, find x as a function of t.

23 The acceleration of a particle P is given by the equation $\frac{d^2x}{dt^2} = 32x\left(x^2+9\right)$, where x metres is the displacement of P from a fixed point O after t seconds. Initially the particle is at O and has a velocity 36 m s^{-1} in the positive direction.

(a) Show that the speed at any position x is given by $4(x^2 + 9)$ m s^{-1}.

(b) Hence find the time taken for the particle to travel 3 metres from O.

24 A body falls from rest so that its velocity v metres per second after t seconds is $v = 80(1 - e^{-0.4t})$.

(a) Show that the acceleration is proportional to $(80 - v)$.

(b) Calculate the distance fallen in the first five seconds.

(c) Calculate the distance fallen when $v = 60$.

25 A particle is brought to top speed by an acceleration that varies linearly with the distance travelled, i.e. $\ddot{x} = kx + C$ where k and C are constants. It starts from rest with an acceleration of 3 m s^{-2} and reaches top speed in a distance of 160 metres. Find:

(a) the top speed **(b)** the speed when the particle has moved 80 metres.

26 The acceleration of a particle P moving in a straight line is given by $a = -x^{-2}$, $x \neq 0$, where x is the distance from O in metres and time is in seconds.

The particle is initially at rest at a point P which is at a distance of 6 metres to the right of O.

The particle moves in a straight line towards O.

(a) Show that $\frac{dx}{dt} = -\sqrt{2}\sqrt{\frac{6-x}{6x}}$.

(b) Using the substitution $x = 6\cos^2\theta$, show that the time taken to reach a distance 3 metres to the right of O can be given by $t = 12\sqrt{3}\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$.

(c) Hence find t, the time taken to reach a distance 3 metres to the right of O, in exact form.

27 A particle is moving in a straight line from a fixed point O on the line, so that at time t seconds it has displacement x metres, a velocity $v\,\text{m s}^{-1}$ and an acceleration $a\,\text{m s}^{-2}$ given by $a = e^{\frac{x}{2}}$.

Initially the particle is at O and moving with a speed of $2\,\text{m s}^{-1}$ while slowing down.

(a) Show that $v = -2e^{\frac{x}{4}}$.

(b) Find an expression for x, v and a in terms of t.

28 A particle has acceleration $a\,\text{m s}^{-2}$ given by $a = v^2 - 3$, where $v\,\text{m s}^{-1}$ is the velocity of the particle when it has a displacement of x metres from the origin.

Find v in terms of x, given that $v = -2$ where $x = 1$.

6.2 MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

Mechanics is a topic that includes both kinematics and dynamics.

- **Kinematics** is the study of the motion of bodies without reference to the causes of their motion.
- **Dynamics** is the study of the effects of forces that cause bodies at rest to move, or that cause bodies in motion to have their state of motion altered.

Forces produce accelerations, so you will be using the principles of kinematics in much of the dynamics material of this chapter.

In this topic, all bodies are treated as particles, which means that all forces are regarded as acting through a single point in the body (as if the body were just a single point). Hence the terms 'particle', 'object' and 'body' will be interchangeable.

Newton's first law of motion

Consider a book resting on the top of your desk. It will remain there in that state of rest unless some external force or forces are applied to change that state. The book is unable to alter its state of rest by itself.

Similarly, consider a marble rolling along a smooth horizontal floor at a constant speed. By itself, this marble is unable to increase or decrease its speed or to change its direction.

This property of bodies is summed up in Newton's first law of motion:

> A body remains at rest or in uniform motion in a straight line unless it is acted on by a non-zero resultant force.

A body can be acted on by several forces that balance each other, so that the resultant force is zero. (For example, a book resting on a horizontal desk is acted on by two forces: the weight force of gravity that acts vertically downwards and the reaction force of the desk that acts vertically upwards.) A zero resultant force is the equivalent of no force acting, so the body remains stationary or in its original state of motion.

Newton's second law of motion

Experience suggests that a given force will produce different accelerations in different bodies. For example, on a smooth horizontal floor the same amount of rolling force applied to a marble and to a heavy steel ball would produce a larger acceleration in the marble. The steel ball is more massive than the marble. Similarly, a piano is more massive than a school desk, as is shown by the fact that it is easier to make the desk move than it is to get the piano to move.

This property of bodies that determines their response to an applied force is called their **inertial mass**. Inertial mass is a measure of a body's resistance to acceleration. This mass is essentially related to the amount of matter that makes up the body. It can be shown experimentally that the ratio of the accelerations produced in two bodies by the same force is the inverse ratio of their masses,

i.e. $\dfrac{a_1}{a_2} = \dfrac{m_2}{m_1}$ so that $m_1a_1 = m_2a_2 =$ a constant, proportional to the same force.

The standard unit of mass is the kilogram (kg). Mass is a scalar quantity.

The **momentum** p of a body is the product of its mass and velocity: $p = mv$

Because the standard unit of mass is the kilogram (kg) and the standard unit of velocity is the metre per second ($m\,s^{-1}$), the standard unit of momentum is the kilogram metre per second ($kg\,m\,s^{-1}$). This is a vector quantity that has the same direction as the velocity.

For example, if a body of mass 5 kg is moving with a velocity of $10\,m\,s^{-1}$, its momentum is $50\,kg\,m\,s^{-1}$. According to Newton's first law of motion, this body is unable by itself to change its velocity, and hence is also unable to change its momentum, unless it is acted on by a non-zero resultant force.

This leads to a statement of Newton's second law of motion:

> The rate of change of momentum is proportional to the applied force and occurs in the direction of the force.

If a body of mass m is acted on by a non-zero resultant force F, then:

$$\begin{aligned} F &\propto \text{rate of change of momentum} \\ &\propto \frac{d}{dt}(mv) \\ &\propto m\frac{dv}{dt} \quad \text{if } m \text{ is constant} \\ \therefore \quad F &\propto ma \quad \text{where } a = \frac{dv}{dt} \end{aligned}$$

That is, $F = kma$ where k is a constant.

By a suitable selection of units, you can make $k = 1$. If 1 unit of force is defined as the amount of force required to produce an acceleration of $1\,m\,s^{-1}$ in a body of mass 1 kg, then: $1 = k \times 1 \times 1 \quad \therefore\ k = 1$

Hence $F = ma$.

This standard unit of force is called a newton (N) where $1\,N = 1\,kg\,m\,s^{-2}$.

Alternative definition for Newton's second law of motion

> The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.

Look at the formula $F = kma$ from the previous definition.

If k and m are constant, then $F \propto a$ and $a \propto F$, which satisfies the alternative definition.

If F and k are constant, then since ma = constant then $a \propto \frac{1}{m}$, which satisfies the alternative definition.

This definition says that 'The acceleration of a body is proportional to the resultant force that acts on the body', which can be written $a \propto F$, or removing the proportionality that $F = Ka$, where K is a constant.

The m, which is a constant in a given situation, is introduced by taking $K = km$, as this makes later calculations easier.

Newton's third law of motion

> When two objects exert force on each other, the forces are equal in magnitude but opposite in direction.
> In other words: For every action there is an equal but opposite reaction.

Dynamics of a particle

You will consider all bodies as particles, so that all external forces acting on a body are regarded as acting through a single point in the body and producing only a translational effect (i.e. no rotation).

You have seen in section 6.1 that acceleration can take different forms apart from $\frac{dv}{dt}$. For example, using the chain rule:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= v\frac{dv}{dx} \qquad \text{as } v = \frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms: $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}, \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

The form to use in a particular problem will depend on the form of the equation that defines acceleration or force:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a = h(v)$, use $\begin{cases} \frac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\frac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

Derivatives with respect to time are often written using dots above the dependent variable, e.g. $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \dot{v} = \frac{dv}{dt}$

Note also that for constant m you have $F = m\frac{dv}{dt}$ and so you obtain: $\frac{dv}{dt} = \frac{F}{m}$

The derivative on the left-hand side is the acceleration, which forms the basis for the solution of a differential equation. As m is constant, the problem is converted to a kinematics problem, for which previous methods can be applied.

Some problems may be set in terms of acceleration and some in terms of force.

Example 8

A force of 5 N is applied to an object of mass 4 kg. Find the acceleration produced in the object.

Solution

Use: $F = ma$

$F = 5, m = 4: \quad 5 = 4a$

$a = 1.25\,\text{ms}^{-2}$

Example 9

A force of 10 N acts on a particle of 25 kg initially at rest. Find:

(a) the acceleration of the particle

(b) the displacement of the particle after 10 seconds.

Solution

(a) Use: $F = ma$

$F = 10, m = 25: \quad 10 = 25a$

$a = 0.4\,\text{ms}^{-2}$

(b) $u = 0,\ a = 0.4,\ t = 10$

$s = ut + \frac{1}{2}at^2$

$s = (0)(10) + \frac{1}{2}(0.4)(10)^2$

$= 0 + 20 = 20$

Displacement is 20 metres.

Example 10

A particle of mass 4 kg is acted on by a force whose direction is constant and whose magnitude at time t seconds is $(12t - 3t^2)$ newtons. If the particle has an initial velocity of 2 m s^{-1} in the direction of the force, find the velocity after 4 seconds.

Solution

There is only one external force acting on the particle, so $F = ma$ becomes:

$$12t - 3t^2 = 4\frac{dv}{dt}$$

where the particle moves with a variable velocity of magnitude $v\text{ m s}^{-1}$ in the direction of the force. Hence:

$$\frac{dv}{dt} = 3t - \frac{3}{4}t^2$$

There are two different methods to complete the solution for the given initial conditions:

Method 1

$$v = \int\left(3t - \frac{3}{4}t^2\right)dt$$

$$= \frac{3}{2}t^2 - \frac{1}{4}t^3 + C$$

When $t = 0$: $v = 2$ and so $C = 2$.

$$\therefore \quad v = \frac{3}{2}t^2 - \frac{1}{4}t^3 + 2$$

When $t = 4$: $v = 10$

Method 2

$$dv = \left(3t - \frac{3}{4}t^2\right)dt$$

$$\int_{\text{initial } v}^{\text{final } v} dv = \int_{\text{initial } t}^{\text{final } t}\left(3t - \frac{3}{4}t^2\right)dt$$

$$\int_2^v dv = \int_0^4\left(3t - \frac{3}{4}t^2\right)dt$$

$$[v]_2^v = \left[\frac{3}{2}t^2 - \frac{1}{4}t^3\right]_0^4$$

$$v - 2 = 24 - 16 - (0 - 0)$$

$$v = 10$$

Hence the velocity after 4 seconds has a magnitude of 10 m s^{-1}.

Example 11

A particle of mass 2 kg moves in a straight line so that at time t seconds its displacement from a fixed origin is x metres and its velocity is $v\text{ m s}^{-1}$. If the resultant force (in newtons) that acts on the particle is:

(a) $6 - 4x$, find v in terms of x given that $v = 2$ when $x = 1$

(b) $8 - 2v^2$, find t in terms of v given that the particle is initially at rest

(c) $8 - 2v^2$, find x in terms of v given that the particle is initially at the origin.

Solution

(a) $F = m\ddot{x}$: $\quad 2\ddot{x} = 6 - 4x$

Hence the equation of motion is $\ddot{x} = 3 - 2x$. As the question requires v in terms of x and the initial conditions are in v and x, you can use either $\ddot{x} = v\dfrac{dv}{dx}$ or $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$:

Method 1

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=3-2x$$

$$\frac{1}{2}v^2=\int(3-2x)\,dx$$

$$\frac{1}{2}v^2=3x-x^2+C$$

When $v=2$, $x=1$

$\therefore\ 2=3-1+C$, so $C=0$:

$$\frac{1}{2}v^2=3x-x^2$$

$$v^2=6x-2x^2$$

$$v=\pm\sqrt{6x-2x^2}$$

Method 2

$$v\frac{dv}{dx}=3-2x$$

$v\,dv=(3-2x)\,dx$

From start $v=2$, $x=1$ to end $v=v$, $x=x$:

$$\int_2^v v\,dv=\int_1^x(3-2x)\,dx$$

$$\left[\frac{v^2}{2}\right]_2^v=\left[3x-x^2\right]_1^x$$

$$\frac{v^2}{2}-2=3x-x^2-(3-1)$$

$$\frac{v^2}{2}=3x-x^2$$

$$v^2=6x-2x^2$$

$$v=\pm\sqrt{6x-2x^2}$$

Which solution for the velocity is valid—positive or negative?

On a quick inspection, you might say that because the initial condition is $v=2$ (i.e. positive), then you should take the positive square root. However, you should recognise from the start of the question that the motion is simple harmonic, as: $\ddot{x}=-2\left(x-\frac{3}{2}\right)$

Hence the particle moves both left *and* right, and so both solutions are valid: $v=\pm\sqrt{6x-2x^2}$

In future examples, Method 2 will be used.

(b) $m\ddot{x}=8-2v^2$ and $m=2$, so the equation of motion is: $\ddot{x}=4-v^2$

You require t in terms of v, hence: $\frac{dv}{dt}=4-v^2$

$$\frac{dt}{dv}=\frac{1}{4-v^2},\qquad v\neq\pm2$$

$$\int_0^t dt=\int_0^v\frac{dv}{4-v^2}$$

Use partial fractions: $\frac{1}{4-v^2}=\frac{1}{(2-v)(2+v)}=\frac{1}{4}\left(\frac{1}{2-v}+\frac{1}{2+v}\right)$

$$[t]_0^t=\frac{1}{4}\left[\log_e\left|\frac{2+v}{2-v}\right|\right]_0^v$$

$$t=\frac{1}{4}\log_e\frac{2+v}{2-v},\qquad -2<v<2$$

(c) $m\ddot{x}=8-2v^2$ and $m=2$, so the equation of motion is: $\ddot{x}=4-v^2$

You require x in terms of v, hence: $v\frac{dv}{dx}=4-v^2$

$$\frac{dv}{dx}=\frac{4-v^2}{v}$$

$$\frac{dx}{dv}=\frac{v}{4-v^2}$$

$$\int_0^x dx=\int_0^v\frac{v}{4-v^2}\,dv\qquad(\text{from start } x=0,\ v=0 \text{ to end } x=x,\ v=v)$$

$$[x]_0^x=-\frac{1}{2}\left[\log_e\left|4-v^2\right|\right]_0^v$$

$$x=-\frac{1}{2}\left(\log_e\left|4-v^2\right|-\log_e 4\right)$$

$$x=\frac{1}{2}\log_e\left|\frac{4}{4-v^2}\right|$$

Example 12

Assume that Earth is a sphere of radius R and that at any point $x \geq R$ distant from the centre of Earth, the acceleration due to gravity is proportional to x^{-2} and is directed towards Earth's centre. Ignore all forces other than Earth's gravity.

A body is projected vertically upwards from the surface of Earth with initial speed V.

(a) Show that the equation of motion of the particle is $\ddot{x} = -\dfrac{gR^2}{x^2}$, where g is the acceleration due to gravity at Earth's surface.

(b) Show that the velocity v of the particle during its flight is given by: $v^2 = V^2 + 2gR^2\left(\dfrac{1}{x} - \dfrac{1}{R}\right)$

(c) Prove that the body's 'escape velocity' is $\sqrt{2gR}$ (i.e. prove that if the particle's initial speed is $V \geq \sqrt{2gR}$, then the particle will escape from Earth and never return).

(d) If $V = \sqrt{2gR}$, prove that the time taken to rise to a height R above Earth's surface is $\dfrac{1}{3}\left(4 - \sqrt{2}\right)\sqrt{\dfrac{R}{g}}$.

Solution

(a) Take O as the centre of Earth and define motion away from O as being in the positive direction.

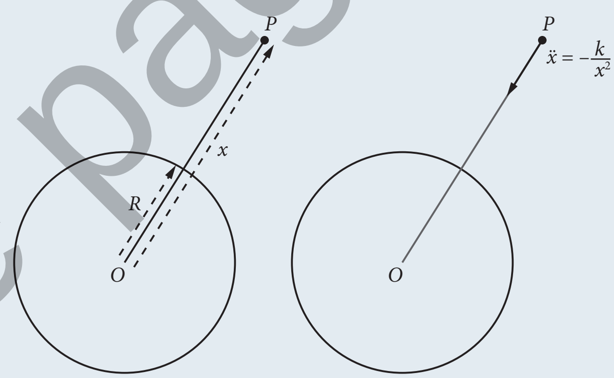
Then: $\ddot{x} = -\dfrac{k}{x^2}$ [1]

But at $x = R$, $\ddot{x} = -g$

(i.e. at the surface, acceleration due to gravity is g):

$$-g = -\frac{k}{R^2} \quad \therefore\ k = gR^2$$

Substitute into [1]: $\ddot{x} = -\dfrac{gR^2}{x^2}$



(b)

$$v\frac{dv}{dx} = -gR^2x^{-2}$$

$$\int_V^v v\,dv = -gR^2\int_R^x x^{-2}\,dx \qquad \text{(from start } v = V, x = R \text{ to end } v = v, x = x\text{)}$$

$$\left[\frac{1}{2}v^2\right]_V^v = gR^2\left[x^{-1}\right]_R^x$$

$$\frac{1}{2}v^2 - \frac{1}{2}V^2 = gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$v^2 = V^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

(c) If the particle escapes, then $x \to \infty$ and so $\dfrac{1}{x} \to 0$

From the solution of part **(b)**: $v^2 \to V^2 + 2gR^2\left(0 - \dfrac{1}{R}\right)$ or $v^2 \to V^2 - 2gR$

But $v^2 \geq 0$ and hence $V^2 - 2gR \geq 0$

i.e. $V^2 \geq 2gR$

As the particle is escaping (i.e. always only moving away from the centre of Earth):

$$V \geq \sqrt{2gR}$$

(d) If $V=\sqrt{2gR}$, the solution of part **(b)** becomes: $v^2=2gR+2gR^2\left(\frac{1}{x}-\frac{1}{R}\right)$

$$\text{i.e. } v^2=\frac{2gR^2}{x}$$

As you are only concerned with motion away from Earth:

$$v=\sqrt{2g}Rx^{-\frac{1}{2}}$$

$$\frac{dx}{dt}=\sqrt{2g}Rx^{-\frac{1}{2}}$$

$$\frac{dt}{dx}=\frac{1}{\sqrt{2g}R}x^{\frac{1}{2}}$$

$$\int_0^t dt=\frac{1}{\sqrt{2g}R}\int_R^{2R}x^{\frac{1}{2}}dx$$

(Note the limits of the integral on the RHS: from Earth's surface to distance R above Earth's surface.)

$$t=\frac{1}{\sqrt{2g}R}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_R^{2R}$$

$$t=\frac{\sqrt{2}}{3R\sqrt{g}}\left(2R\sqrt{2R}-R\sqrt{R}\right)$$

$$t=\frac{\sqrt{2}}{3R\sqrt{g}}\left(2\sqrt{2}-1\right)R\sqrt{R}$$

$$t=\frac{\left(4-\sqrt{2}\right)R\sqrt{R}}{3R\sqrt{g}}$$

$$t=\frac{1}{3}\left(4-\sqrt{2}\right)\sqrt{\frac{R}{g}}$$

Example 13

A particle of mass 1 kg is fixed in position, being suspended from the ceiling by two light rods AP and BP. The lengths of AP and BP are in the ratio 3 : 2, and AP is inclined at an angle of 30° below the ceiling. The tension forces in the rods AP and BP are T_1 and T_2 respectively.

(a) Show that the rod BP is inclined at an angle α below the ceiling, where $\sin\alpha=\frac{3}{4}$.

(b) Draw a diagram to show the three forces (T_1, T_2 and the weight force) that act on the particle.

(c) Resolve the forces into horizontal and vertical components.

(d) Use Newton's first law of motion to calculate the values of T_1 and T_2 correct to 1 decimal place. (Use $g=9.8\text{ m s}^{-2}$.)

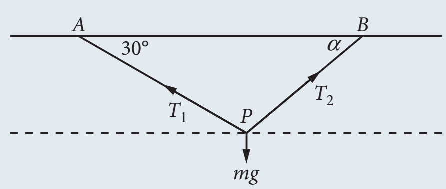
Solution

(a) Using the sine rule in triangle ABP:

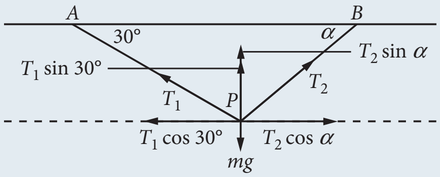
$$\frac{\sin\alpha}{3}=\frac{\sin 30^\circ}{2}$$

$$\sin\alpha=\frac{3}{4}$$

(b)



(c)



(d) The particle is fixed in position, so according to Newton's first law of motion there is a zero resultant force acting horizontally and a zero resultant force acting vertically.

Horizontally: $T_1 \cos 30° = T_2 \cos \alpha$

$\sin\alpha = \frac{3}{4} \therefore \cos\alpha = \frac{\sqrt{7}}{4}$: $\frac{\sqrt{3}}{2}T_1 = \frac{\sqrt{7}}{4}T_2$ [1]

Vertically (mass $m = 1$): $T_1 \sin 30° + T_2 \sin \alpha = 9.8$

$\frac{1}{2}T_1 + \frac{3}{4}T_2 = 9.8$ [2]

From [1]: $T_1 = \frac{\sqrt{21}}{6}T_2$

Substitute into [2]: $\frac{\sqrt{21}}{12}T_2 + \frac{3}{4}T_2 = 9.8$

$T_2\left(\sqrt{21} + 9\right) = 117.6$

$T_2 = 8.65815\ldots$

Substitute into [1]: $T_1 = 6.61277\ldots$

$\therefore$ Correct to 1 decimal place: $T_1 = 6.6$ and $T_2 = 8.7$

Resultant force

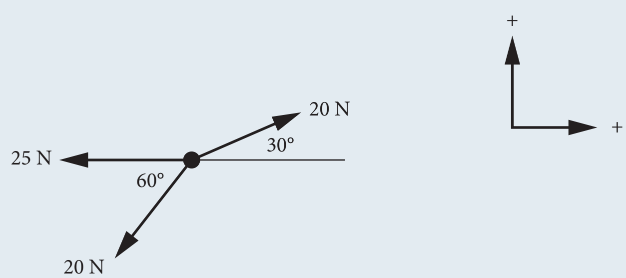
If the forces acting on a body (particle) are not in equilibrium, then there is a resultant force, or net force, on the body.

Example 14

The diagram shows the forces acting on a body.
Find the resultant force,

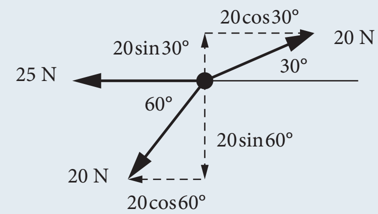
(a) giving its strength and direction:

(b) in vector form.



Solution

(a) Resolve the forces into vertical and horizontal components.



Horizontally: Left, 25 N and $20\cos 60° = 10$ N

Right: $20\cos 30° = 10\sqrt{3}$ N

Resultant force: $10\sqrt{3} - 25 - 10 = 10\sqrt{3} - 35$ N

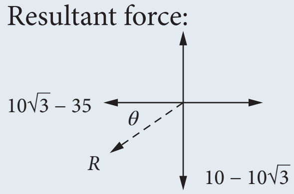
Vertically: Upwards: $20\sin 30° = 10$ N

Downwards: $20\sin 60° = 10\sqrt{3}$ N

Resultant force: $10 - 10\sqrt{3}$ N

Since both the resultant forces are negative the effective force is downwards to the left.

Resultant force:



By Pythagoras:

$$R = \sqrt{\left(10\sqrt{3} - 35\right)^2 + \left(10 - 10\sqrt{3}\right)^2}$$

$$= \sqrt{300 - 700\sqrt{3} + 1225 + 100 - 200\sqrt{3} + 300} = \sqrt{1925 - 900\sqrt{3}} \approx 19.1\text{ N}$$

$$\tan\theta = \frac{10 - 10\sqrt{3}}{10\sqrt{3} - 35} = 0.41407$$

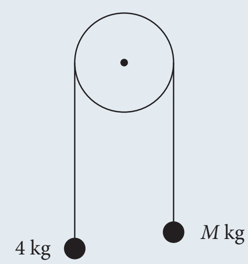
$$\theta = 22°30'$$

The resulting force is 19.1 N downwards at an angle 22°30′ to the left direction.

(b) $R = \left(10\sqrt{3} - 35\right)\underset{\sim}{i} + (10 - 10\sqrt{3})\underset{\sim}{j}$

Example 15

The diagram shows objects of mass 4 kg and M kg attached to the ends of a light inextensible string that passes over a smooth pulley. The 4 kg object is accelerating upwards at a rate of 4.9 m s^{-2}. Let the tension in the string be T newtons. Find the value of T and hence determine the value of M.



Solution

$g = 9.8$ m s^{-2}

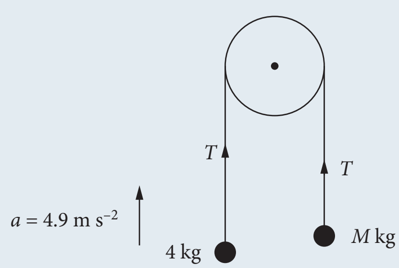
$T - 4g = 4 \times 4.9$

$T = 19.6 + 4g = 58.8$ N

$Mg - T = 4.9M$

$M(g - 4.9) = 58.8$

$M = \dfrac{58.8}{9.8 - 4.9} = \dfrac{58.8}{4.9} = 12$ kg



EXERCISE 6.2 MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

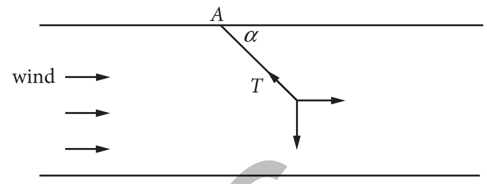
1 A force of 10 N is applied to an object of mass 8 kg. Find the acceleration produced in the object.

2 A force of 5 N acts on a particle of 10 kg initially at rest. Find:

(a) the acceleration of the particle

(b) the displacement of the particle after 20 seconds.

3 A 1 kg object in a wind tunnel is suspended from a point A on the ceiling by a light rope. A wind equivalent to a horizontal force of 4.9 N is directed down the tunnel. This causes the rope and the object to move from their position vertically below A to be inclined at an angle α to the ceiling. The object remains in this position while the wind is blowing.



The diagram shows the three forces (tension T newtons in the rope, the wind force and the weight force) acting on the object. Resolve the forces into horizontal and vertical components. Noting that the object is stationary, use Newton's first law of motion to find the values of α and T (Use $g = 9.8\ \text{m s}^{-2}$). The correct equations are:

A $\alpha = \tan^{-1} 2$ and $T = \dfrac{2g}{\sqrt{5}}$

B $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{2g}{\sqrt{5}}$

C $\alpha = \tan^{-1} 2$ and $T = \dfrac{\sqrt{5}g}{2}$

D $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$ and $T = \dfrac{\sqrt{5}g}{2}$

4 A particle of mass 1 kg moves in a straight line such that at time t seconds its displacement from a fixed origin is x metres and its velocity is $v\ \text{m s}^{-1}$. If the resultant force is $3 + 2t$ (in newtons), find its displacement in terms of t given that $v = 1$ and $x = 2$ when $t = 0$.

5 The velocity $v\ \text{m s}^{-1}$ of a particle of mass 2 kg that is moving in a straight line is $v = t^2 - 6t + 8$ at any time t s.

(a) Find the resultant force in terms of t.

(b) Find the force when the particle is stationary.

6 A particle of mass 10 kg starts from rest at a point A and moves in a straight line under the action of a force that decreases uniformly from 20 N to zero in 20 seconds. The particle then travels with constant velocity for a further 20 seconds. After this, the particle moves under the action of a retarding force of 40 N until it comes to rest at point B.

(a) Express the force that applies for the first 20 seconds as a function of t.

(b) Find the velocity of the particle at $t = 20$ seconds.

(c) Find the time taken for the retarding force to stop the motion of the particle.

(d) Find the maximum speed attained during the motion.

(e) Find the total distance travelled during the motion.

(f) Find the average speed during the motion.

7 A particle of mass 10 kg moves under the action of a force so that its velocity $v\ \text{m s}^{-1}$ is given by $v = \sqrt{x^2 - 6x + 5}$. Find the force F in terms of the displacement x.

8 A particle of mass m moves so that its velocity v is given by $v = f(x)$. The resultant force that causes this motion is given by:

A $mf'(x)$

B $mxf'(x)$

C $mf'\left(\dfrac{1}{2}x^2\right)$

D $mf(x)f'(x)$

9 A particle of mass 8 kg is acted on by a force whose magnitude is $2(25 + 60x - 6x^2)$ newtons, where x is the displacement from a fixed point O. If the particle is initially at rest at O, find its speed when it is 10 metres from O.

10 A particle of mass 2 kg moves in a straight line. At time t, its displacement from a fixed origin is x metres and its velocity is v m s^{-1}. If the resultant force (in newtons) acting on the particle is:

(a) $6\cos t$, and $v = 2$ and $x = 0$ when $t = 0$, then find x in terms of t

(b) $2 + 4x$, and $v = 2$ when $x = 0$, then find v when $x = 2$

(c) $4 - 2v$, and $v = 0$ when $t = 0$, then find the time when $v = 1$.

11 At time t a particle of mass m is moving in a straight line under the action of a force given by $F = \frac{m(3-5x)}{x^3}$. The particle starts from rest at $x = \frac{1}{3}$.

(a) Find its velocity in terms of x.

(b) At what other point, if any, does the particle come to rest?

12 A particle of unit mass is acted on by a force $F = v^2 \log_e v$, where v m s^{-1} is the velocity of the particle. The motion starts from O with a velocity of e m s^{-1}. Find the displacement when the velocity is e^2 m s^{-1}.

13 The acceleration of an object moving towards a planet under gravitational attraction varies inversely as the square of the distance from the centre of the planet (i.e. $\ddot{x} = -\frac{k}{x^2}$ where x is the displacement from the centre of the planet and k is a constant). Show that if the object starts from rest at a distance a from the centre of the planet, its speed at distance x from the centre of the planet is: $\sqrt{\frac{2k(a-x)}{ax}}$

14 A particle of unit mass starts from rest with displacement b (where $b > 0$) and is attracted towards the origin O with an acceleration of magnitude $\frac{k}{x^2}$ where x is the displacement from the origin and k is a positive constant.

(a) Explain why $\ddot{x} = -\frac{k}{x^2}$.

(b) Show that the velocity v is given by: $v^2 = 2k\left(\frac{1}{x} - \frac{1}{b}\right)$

(c) Use the substitution $x = b\cos^2\theta$ to show that: $\int \sqrt{\frac{x}{b-x}}\,dx = -\sqrt{bx - x^2} - \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) + C$

(d) Hence show that the time required for the particle to reach the origin is: $\pi\left(\frac{b^3}{8k}\right)^{\frac{1}{2}}$

15 An object of mass m moves in a straight line under a force of magnitude $mk^2\left(x + \frac{a^4}{x^3}\right)$, $k > 0$ towards the origin O. If the particle started from rest at a distance a units from O, show that its speed when $x = \frac{a}{2}$ is $\frac{\sqrt{15}ka}{2}$.

16 A vertical pole subtends an angle α at a point P in the same horizontal plane as the foot of the pole. Two particles are projected at the same instant from P in directions that make angles α_1 and α_2 with the horizontal, with initial speeds v_1 and v_2, so that the first particle hits the top of the pole at the same instant that the second particle hits the bottom.

(a) Show that: $v_1\cos\alpha_1 = v_2\cos\alpha_2$

(b) Show that the time of flight is: $\frac{2v_2\sin\alpha_2}{g}$

(c) Hence prove that: $\tan\alpha = \tan\alpha_1 - \tan\alpha_2$

17 A projectile A is projected from O with speed u at an angle α above the horizontal. A package B is parachuting vertically downwards at a constant speed equal to $-u\sin\alpha$. At the moment when A is projected, the package B is at the point $Q\left(\frac{u^2}{g}\sin 2\alpha, \frac{2u^2}{g}\sin^2\alpha\right)$.

(a) Find the coordinates of A and B at time t after A is projected.

(b) Show that a searchlight, which is located at O and which is moved so that its beam is continually directed at A, will always have B in its beam.

18 Two stones are thrown simultaneously from the same point in the same direction and with the same non-zero angle of projection α (upward inclination to the horizontal), but with different velocities u, v metres per second ($u < v$).

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant, the faster stone just clears a wall ZQ of height h metres above the level of projection as its (downward) path makes an angle β with the horizontal.

(a) Show that while both stones are in flight, the line joining them has an inclination to the horizontal which is independent of time.

(b) Hence express the horizontal distance from P to Z (the foot of the wall) in terms of h and α.

(c) Show that $v(\tan\alpha + \tan\beta) = 2u\tan\alpha$.

(d) Deduce that if $\beta = \frac{1}{2}\alpha$ then $u < \frac{3}{4}v$.

19 A ball thrown from a point A with speed V at an inclination α to the horizontal reaches a point B after t seconds.

(a) Find the position of B relative to A.

(b) Show that if AB is inclined at θ to the horizontal then the direction of motion of the ball when at B is inclined to the horizontal at an angle β, given by $\tan\beta = 2\tan\theta - \tan\alpha$.

20 A particle is projected, with speed V and inclination α above the horizontal, from a point O on a plane inclined at 30° to the horizontal (so that $\alpha > 30°$). It lands at a point Q on the inclined plane, at right angles to the plane.

(a) Find the coordinates of Q.

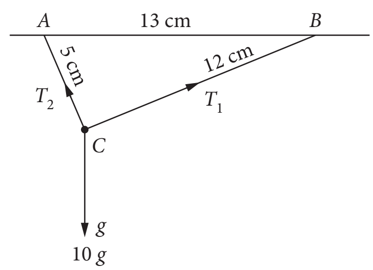
(b) Find the horizontal and vertical components of the velocity when the particle is at Q.

(c) Show that the time of flight is: $t = \dfrac{V\left(\sin\alpha + \sqrt{3}\cos\alpha\right)}{g}$

(d) Show that: $\tan\alpha = \dfrac{5\sqrt{3}}{3}$

(e) Hence show that the range on the inclined plane (i.e. OQ) is: $\dfrac{4V^2}{7g}$

21 A particle of mass 10 kg is suspended by two strings, at point C, of length 5 cm and 12 cm respectively, that are attached to two fixed points, A and B, 13 cm apart on the same horizontal level, as shown in the diagram.



(a) Show that $\angle ACB = 90°$.

(b) Calculate the magnitudes of $\angle ABC$ and $\angle BAC$.

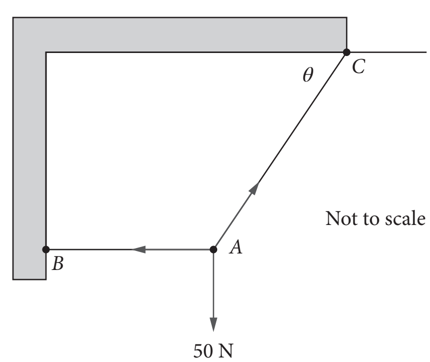
(c) By resolving the forces horizontally and vertically, calculate the magnitudes of T_1 and T_2.

22 Three forces act on an object which moves with constant velocity $\underset{\sim}{v} = \left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)\ \text{ms}^{-1}$.

Two of the forces are $\left(3\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k}\right)$ newtons and $\left(4\underset{\sim}{i} - 7\underset{\sim}{j} + 2\underset{\sim}{k}\right)$ netwons. Find the third force.

23 A particle A with a tension of 50 N is held in equilibrium by two strings, AB and AC.

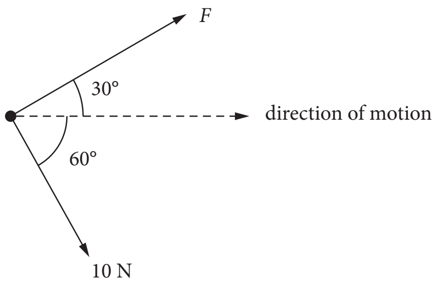
AB is attached to the wall and is horizontal. AC is attached to the ceiling and makes an angle of θ with the horizontal, as shown in the diagram. The tension in AC is twice the tension in AB.



(a) Find the value of θ.

(b) Calculate the tension in the strings AB and AC. Answer correct to one decimal place.

24 An object with a mass of 2 kg is acted upon by forces of 10 N and F N at 60° and 30° respectively to the direction of motion as shown in the diagram. Find the acceleration of the object.

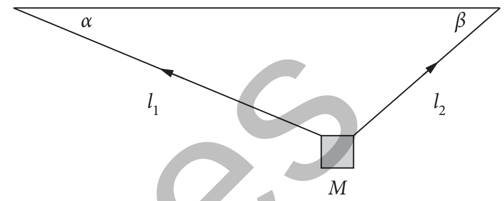


25 A body of mass M kg is being suspended by two strings of differing lengths l_1 and l_2 attached to the mass and then to either end of a horizontal rod as shown in the diagram.

Acceleration due to gravity is g m s^{-2}.

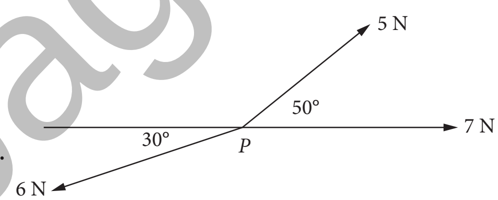
The tensions in l_1 and l_2 are T_1 and T_2 (newtons) respectively.

Show that the tension T_1 in string l_1 is given by $T_1 = \dfrac{Mg\cos\beta}{\sin(\alpha+\beta)}$.



26 Three coplanar forces act at a point P. The magnitudes are 5 N, 6 N and 7 N.

The directions in which the forces act are shown in the diagram. Find the magnitude and direction of the resultant of the three forces. Give the magnitude correct to four significant figures and the direction correct to the nearest degree.



27 The diagram shows two objects with masses of m and $4m$ kg on either end of a light inextensible string that passes through a smooth pulley. Both objects are released from rest simultaneously.

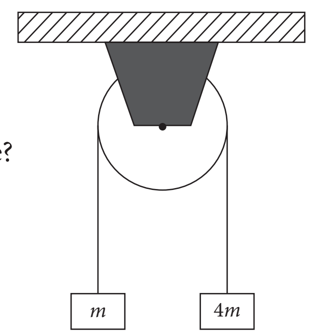
Let g be the acceleration due to gravity. After 4 seconds, which of the following is true?

A The heavier object has a speed of $\frac{3g}{5}$ m s^{-1}.

B The heavier object has travelled $\frac{24g}{5}$ metres.

C The heavier object has an acceleration of $\frac{3g}{4}$ m s^{-2}.

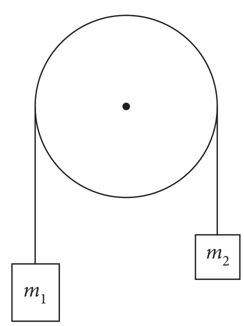
D The heavier object has stopped moving as the lighter has hit the pulley.



28 Two bodies of masses m_1 and m_2 are connected by a light inextensible string and pass over a smooth pulley. If the mass m_1 is coming down, what is the acceleration of the mass m_2?

A $\dfrac{g(m_1+m_2)}{m_1-m_2}$ **B** $\dfrac{g(m_1-m_2)}{m_1+m_2}$

C $\dfrac{g(m_1m_2)}{m_1-m_2}$ **D** $\dfrac{g(m_1m_2)}{m_1+m_2}$



6.3 SIMPLE HARMONIC MOTION (SHM)

Consider the motion of a particle moving along a straight line back and forth, so that its displacement from a fixed point at time t is given by a sine or cosine function.

Example 16

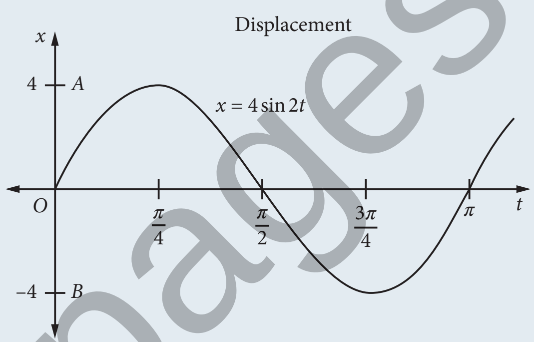
A particle moves in a straight line so that its displacement x m from a fixed point O at time t seconds is defined by $x = 4\sin 2t$. After considering properties of the graph of $x = 4\sin 2t$ to analyse the motion of the particle, find expressions for:

(a) the velocity **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

Look at the properties of the sine curve to help determine the nature of the movement of the particle.

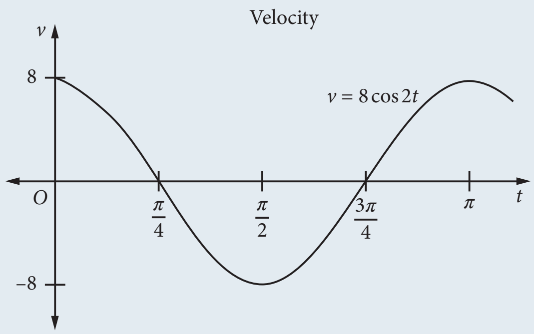
- When $t = 0$, $x = 0$
- When $t = \frac{\pi}{4}$, $x = 4$
- When $t = \frac{\pi}{2}$, $x = 0$
- When $t = \frac{3\pi}{4}$, $x = -4$
- When $t = \pi$, $x = 0$



Remember that the particle is moving in a straight line along the x-axis. It starts at $x = 0$ and takes $\frac{\pi}{4}$ seconds (approximately 0.8 s) to move to A, another $\frac{\pi}{4}$ seconds to return to O, another $\frac{\pi}{4}$ seconds to move to B and then another $\frac{\pi}{4}$ seconds to return to O again. This pattern of movement repeats every π seconds. You say that the particle **oscillates** from A to B about the point O, the centre of the motion, with a **period** of π. The distance between O and the extreme positions A and B (in this case 4 m) is called the **amplitude**.

(a) $v = \frac{dx}{dt} = 8\cos 2t$ [1]

- When $t = 0$, $v = 8$ at $x = 0$
- When $t = \frac{\pi}{4}$, $v = 0$ at $x = 4$
- When $t = \frac{\pi}{2}$, $v = -8$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $v = 0$ at $x = -4$
- When $t = \pi$, $v = 8$ at $x = 0$



In the original displacement diagram above, the particle is at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ at $x = 4$ and when $t = \frac{3\pi}{4}$ at $x = -4$.

The velocity diagram above right shows that $v = 0$ when $t = \frac{\pi}{4}$ and when $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 0$ and $v = 8$. When $t = \frac{\pi}{2}$, $x = 0$ and $v = -8$. This means that at O, $v = 8$ when the particle is travelling in the positive direction (towards A) but $v = -8$ when the particle is travelling in the negative direction (towards B).

Squaring [1]: $v^2 = 64\cos^2 2t$

$$v^2 = 64(1 - \sin^2 2t)$$

$$v^2 = 64\left(1 - \frac{x^2}{16}\right)$$

$$v^2 = 4(16 - x^2) \qquad [2]$$

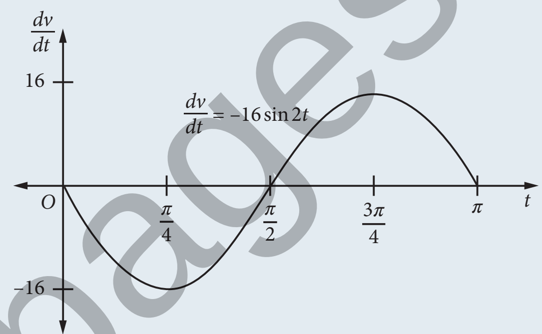
Equation [2] gives the velocity in terms of x.

(b) $\ddot{x} = \dfrac{dv}{dt} = -16\sin 2t \qquad [3]$

$\ddot{x} = -4x \qquad [4]$

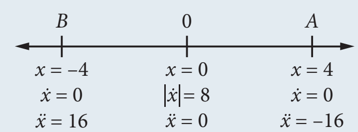
Equation [3] gives the acceleration in terms of t. Equation [4] gives the acceleration in terms of x. $\ddot{x} = -4x$ tells us that when $x = 0$, $\ddot{x} = 0$; when $x = 4$ (at A), $\ddot{x} = -16$; and when $x = -4$ (at B), $\ddot{x} = 16$. $\ddot{x}$ and x always have the opposite sign.

- When $t = 0$, $\ddot{x} = 0$ at $x = 0$
- When $t = \dfrac{\pi}{4}$, $\ddot{x} = -16$ at $x = 4$
- When $t = \dfrac{\pi}{2}$, $\ddot{x} = 0$ at $x = 0$
- When $t = \dfrac{3\pi}{4}$, $\ddot{x} = 16$ at $x = -4$
- When $t = \pi$, $\ddot{x} = 0$ at $x = 0$



(c) In this type of motion, the particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and the magnitude of this acceleration is proportional to its distance from the fixed point.

Taking O as the fixed point, the above description tells you that when the particle is at A ($x > 0$) its acceleration is towards O ($\ddot{x} < 0$), while when the particle is at B ($x < 0$) its acceleration is again towards O ($\ddot{x} > 0$).



Note that the equation of motion $x = 4\sin 2t$ in this example could also be written as $x = 4\cos\left(2t - \dfrac{\pi}{2}\right)$.

In a situation such as Example **16**, $\ddot{x}$ and x are always opposite in sign and the magnitude of the acceleration is always proportional to the distance from O. This means you can take the differential equation $\ddot{x} = -4x$ from this example and write it as a general equation $\ddot{x} = -kx$, where k can be any positive constant, to define all motion of this type.

This type of motion is called **simple harmonic motion** (SHM) and it can be applied to many real-life physical situations. Because k is a positive constant, it is usually replaced by n^2 so that $\ddot{x} = -n^2 x$ is the basic equation of SHM.

In general, for simple harmonic motion you have:

		OR	
• displacement x:	$x = A\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$		$x = A\sin(nt + \alpha)$, $\alpha > 0$, $n > 0$
• velocity, $\dot{x}$:	$\dot{x} = -An\sin(nt + \alpha)$		$\dot{x} = An\cos(nt + \alpha)$
• acceleration $\ddot{x}$:	$\ddot{x} = -An^2\cos(nt + \alpha)$		$\ddot{x} = -An^2\sin(nt + \alpha)$
or	$\ddot{x} = -n^2 x$		$\ddot{x} = -n^2 x$
• Squaring the velocity:	$v = -An\sin(nt + \alpha)$		$v = -An\cos(nt + \alpha)$
	$v^2 = A^2n^2\sin^2(nt + \alpha)$		$v^2 = A^2n^2\cos^2(nt + \alpha)$
	$= n^2\left(A^2 - A^2\cos^2(nt + \alpha)\right)$		$= n^2\left(A^2 - A^2\sin^2(nt + \alpha)\right)$
	$= n^2\left(A^2 - x^2\right)$		$= n^2\left(A^2 - x^2\right)$

So, regardless of your starting point you have the very useful result: $v^2 = n^2(A^2 - x^2)$.

As you work through this chapter, you will discover which form (sine or cosine) is best to use in a particular situation.

Of course, if you start with the differential equation $\ddot{x} = -n^2 x$ and use either form, the forms will only differ by a constant, which will be given by the initial conditions.

Important results

1. When $\dot{x} = 0$ (e.g. at A and B in Example **16**) the magnitude of the acceleration is greatest.
2. When $\ddot{x} = 0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. the velocity has its greatest or least value).

The general equation $x = A\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$

If when $t = 0$, $x = A$, the particle is initially at A (the extreme point) and $A = A\cos\alpha$, then $\cos\alpha = 1$ and $\alpha = 0$. The equation of motion can then be written $x = A\cos nt$.

$$x = A\cos nt \qquad [1]$$

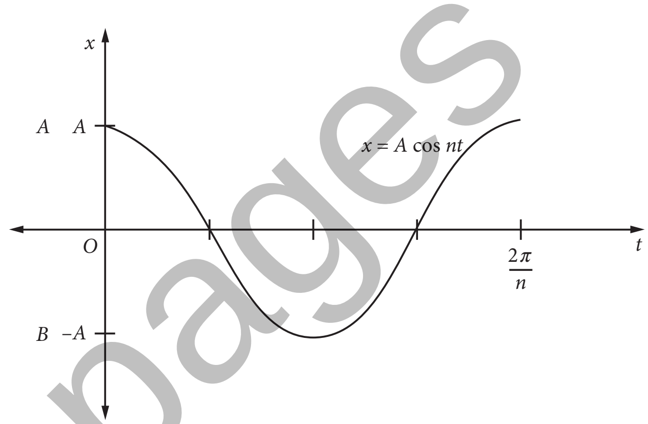
$$v = \dot{x} = -nA\sin nt \qquad [2]$$

$$\begin{aligned} v^2 &= n^2A^2\sin^2 nt \\ &= n^2A^2(1 - \cos^2 nt) \\ &= n^2A^2\left(1 - \frac{x^2}{A^2}\right) \end{aligned}$$

$$v^2 = n^2(A^2 - x^2) \qquad [3]$$

$$\frac{dv}{dt} = \ddot{x} = -n^2A\cos nt$$

$$\ddot{x} = -n^2 x \qquad [4]$$



- The period T is the time for one complete oscillation: $T = \dfrac{2\pi}{n}$
- The frequency f is the number of oscillations per unit time: $f = \dfrac{1}{T} = \dfrac{n}{2\pi}$
- The amplitude A is the distance from the centre of motion O to either of the extreme points A or B.

Simple harmonic motion problems are usually solved using either $x = A\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(A^2 - x^2)$. The starting point will depend on the information given.

The general equation $x = A\cos(nt + a) + c$ can be rewritten as $x - c = A\cos(nt + a)$, $a > 0$, $n > 0$, so the addition of the c to the equation results in the centre of the motion now being $x = c$.

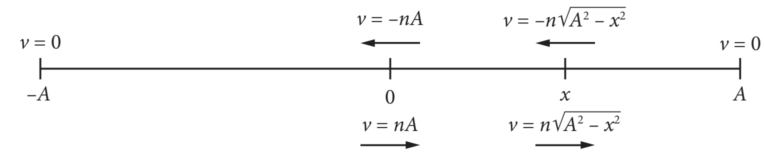
Simple harmonic motion (SHM)—summary

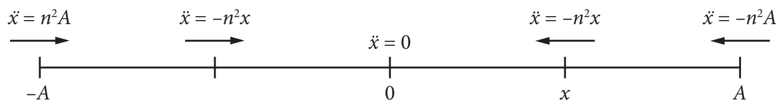
$x = A\cos(nt + \alpha)$ $= A\cos nt$ if $x(0) = a$ $v = \dot{x} = -nA\sin nt$	$\dfrac{dv}{dt} = \ddot{x} = -n^2A\cos nt$ $\ddot{x} = -n^2x$	$v^2 = n^2(A^2 - x^2)$ $-A \le x \le A$ $T = \dfrac{2\pi}{n} = \dfrac{1}{f}$

Note:

- $x = A\sin(nt + \alpha)$ can also describe the displacement function, but it is more conventional to use $x = A\cos(nt + \alpha)$.
- In problems involving a pendulum, the motion usually starts at the maximum displacement, so $\alpha = 0$ and the equation of motion becomes $x = A\cos nt$.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.





Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The magnitude of the force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

MAKING CONNECTIONS

Simple harmonic motion

Use technology to explore the representation of a point moving along a straight line as a trigonometric function.

Example 17

The displacement x m of a particle moving in a straight line is given by $x = 4\cos 6t$. Discuss the motion of the particle.

Solution

$x = 4\cos 6t$ is of the form $x = A\cos(nt + \alpha)$, so the motion is simple harmonic about the origin.

Amplitude $A = 4$ $\quad n = 6 \quad$ Period $= \dfrac{2\pi}{n} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$

When $t = 0$: $x = 4\cos 0 = 4$

$\therefore$ The particle starts 4 m to the right of O.

Velocity: $\dot{x} = -24\sin 6t$

When $t = 0$: $\dot{x} = -24\sin 0 = 0$

$\therefore$ The particle is initially at rest.

Acceleration: $\ddot{x} = -144\cos 6t$

$\ddot{x} = -36x$

When $t = 0$: $\ddot{x} = -144$

$\therefore$ The initial acceleration is 144 m s^{-2} towards O.

The motion is simple harmonic with amplitude 4 m, period $\dfrac{\pi}{3}$ seconds, initially at rest 4 m to the right of O with an acceleration of 144 m s^{-2} towards O.

Example 18

The motion of a particle moving along a straight line is given by the equation $\frac{d^2x}{dt^2} = -16x$.

If $x = 0$ and $v = 4$ when $t = 0$, find its displacement at any time t and state the period and amplitude.

Solution

Because $\frac{d^2x}{dt^2} = -16x = -n^2x$ where $n = 4$, the motion is simple harmonic.

$$\therefore \quad x = A\cos(nt + \alpha)$$

$$\text{For } n = 4: \quad x = A\cos(4t + \alpha) \qquad [1]$$

$$v = \dot{x} = -4A\sin(4t + \alpha) \qquad [2]$$

When $t = 0$, $x = 0$ in [1]: $0 = A\cos\alpha$

$$\alpha = \frac{\pi}{2}$$

When $t = 0$, $v = 4$ in [2]: $4 = -4A\sin\frac{\pi}{2}$

$$A = -1$$

$$\therefore \quad x = -\cos\left(4t + \frac{\pi}{2}\right)$$

You can use the identity $-\cos\theta = \cos(\pi - \theta)$ to remove the negative sign and write the answer in a more familiar form:

$$-\cos\left(4t + \frac{\pi}{2}\right) = \cos\left(\pi - 4t - \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2} - 4t\right)$$

$$= \cos\left(4t - \frac{\pi}{2}\right)$$

$$\therefore \quad x = \cos\left(4t - \frac{\pi}{2}\right) \qquad \text{Period} = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \qquad \text{Amplitude} = A = 1$$

It is worth noticing that $\cos\left(\frac{\pi}{2} - 4t\right) = \sin 4t$, so the equation of motion could be written as $x = \sin 4t$.

This would make it easier to sketch the function.

Example 19

A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x} = -4x$. Find its period, amplitude and displacement at time t given that at $t = 0$, $x = 3$ and $v = -6\sqrt{3}$.

Solution

$\ddot{x} = -4x = -n^2 x$ where $n = 2$, so the motion is simple harmonic.

$$x = A\cos(nt + \alpha)$$

For $n = 2$: $\quad x = A\cos(2t + \alpha)$

When $t = 0$, $x = 3$: $\quad 3 = A\cos\alpha \qquad [1]$

Velocity: $\quad \dot{x} = -2A\sin(2t + \alpha)$

When $t = 0$, $v = -6\sqrt{3}$: $\quad -6\sqrt{3} = -2A\sin\alpha$

$$3\sqrt{3} = A\sin\alpha \qquad [2]$$

[2] ÷ [1]: $\quad \dfrac{A\sin\alpha}{A\cos\alpha} = \dfrac{3\sqrt{3}}{3}$

$$\tan\alpha = \sqrt{3}$$

As $A > 0$: $\sin\alpha > 0$ (from [2]) and $\cos\alpha > 0$ (from [1]), so α is in the first quadrant and is an acute angle.

Hence: $\quad \alpha = \dfrac{\pi}{3}$

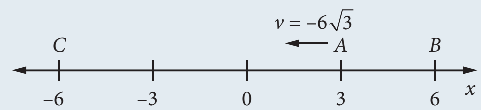
From [1]: $\quad 3 = A\cos\dfrac{\pi}{3}$

$$A = 6$$

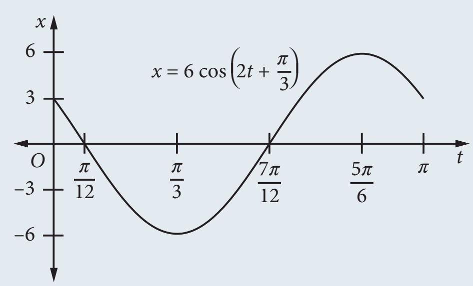
$$\therefore \quad x = 6\cos\left(2t + \frac{\pi}{3}\right)$$

Hence the period $= \dfrac{2\pi}{2} = \pi$, amplitude $= 6$ and displacement is given by $x = 6\cos\left(2t + \dfrac{\pi}{3}\right)$.

This diagram shows the motion of the particle at $t = 0$.
It starts at A and moves towards the centre of the motion.
C and B are the extreme points of the motion.



This diagram shows the displacement x for any time t in the domain $0 \le t \le \pi$.



Example 20

The speed v m s^{-1} of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where the magnitude of its displacement from a fixed point O is x m. Show that the motion is simple harmonic and find:

(a) the centre of the motion **(b)** the period **(c)** the amplitude.

Solution

$v^2 = f(x)$, so it seems likely that using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ might help.

$$v^2 = 6 + 4x - 2x^2$$

$$\frac{1}{2}v^2 = 3 + 2x - x^2$$

Differentiate with respect to x: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 2x$

Hence: $\ddot{x} = -2(x - 1)$

With $y = x - 1$: $\ddot{y} = -2y$ (as $\ddot{y} = \ddot{x}$)

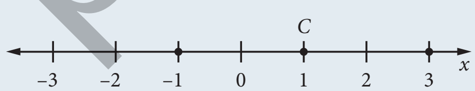
This is simple harmonic motion about $y = 0$.

(a) With $y = x - 1$, SHM about $y = 0$ is the same as SHM about $x = 1$.

(b) $n^2 = 2$, so $n = \sqrt{2}$ and the period $= \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.

(c) The extreme positions of the particle are found where $v = 0$.

$$6 + 4x - 2x^2 = 0$$
$$-2(x^2 - 2x - 3) = 0$$
$$(x + 1)(x - 3) = 0$$



$$x = -1 \text{ or } 3$$

The particle oscillates between $x = -1$ and $x = 3$ about the centre $x = 1$, so the amplitude is 2 m.

Example 21

A particle moves in a straight line so that its position at any time t is given by $x = 3\cos 2t + 4\sin 2t$.

(a) Show that $3\cos 2t + 4\sin 2t = 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$.

(b) Show that the motion is simple harmonic and find its greatest speed in metres per second.

Solution

(a) Expression $= 3\cos 2t + 4\sin 2t$

$\sqrt{3^2 + 4^2} = 5$: $= 5\left(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right)$

$\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$: $= 5(\cos 2t\cos\alpha + \sin 2t\sin\alpha)$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}$: $= 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$

(b) $$x = 5\cos(2t - \alpha)$$

Velocity: $\dot{x} = -10\sin(2t - \alpha)$

Acceleration: $\ddot{x} = -20\cos(2t - \alpha)$

$= -4x$

$\ddot{x} = -n^2 x$ with $n = 2$, so the motion is simple harmonic.

The greatest speed occurs when $\ddot{x} = 0$, i.e. when $\cos(2t - \alpha) = 0$.

Hence: $2t - \alpha = \dfrac{\pi}{2}$

Thus: $\dot{x} = -10\sin\dfrac{\pi}{2}$

$= -10$ The greatest speed is $10\,\text{m s}^{-1}$.

Example 22

A particle moving in a straight line with SHM has a speed of $15\,\text{m s}^{-1}$ when passing through its mean position (the centre of the motion). Find the amplitude of the motion and the acceleration in the extreme positions, given that the period of the motion is 2 seconds.

Solution

$x = 0 \quad |v| = 15 \quad T = 2 = \dfrac{2\pi}{n}$ hence $n = \pi$.

The values of x, v and n are known. Use the result $v^2 = n^2(A^2 - x^2)$ to find A.

$$225 = \pi^2(A^2 - 0)$$

$$A^2 = \frac{225}{\pi^2}$$

$$A = \frac{15}{\pi} \quad \text{as } A > 0$$

Motion is SHM, so: $\ddot{x} = -n^2 x$

Hence: $\ddot{x} = -\pi^2 x$

At the extreme position, $x = A = \dfrac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \dfrac{15}{\pi} = -15\pi$

At the other extreme position, $x = -\dfrac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \left(-\dfrac{15}{\pi}\right) = 15\pi$

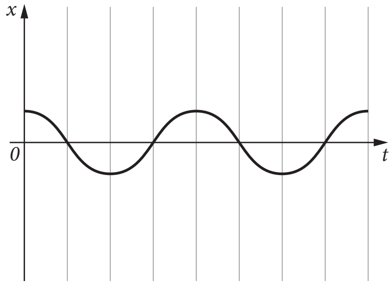
Thus the amplitude of the motion is $\dfrac{15}{\pi}$ metres and the magnitude of the acceleration at the extremes is $15\pi\,\text{m s}^{-2}$.

Using graphs for displacement, velocity and acceleration

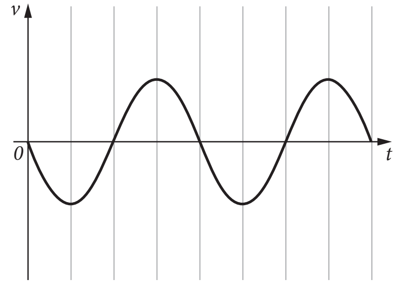
Consider a particle moving in a straight line with simple harmonic motion where the displacement is given by $x = 3\cos 2x$ for $0 \le t \le 2\pi$.

Find the expressions for velocity and acceleration and graph each of the three equations on the same timescale.

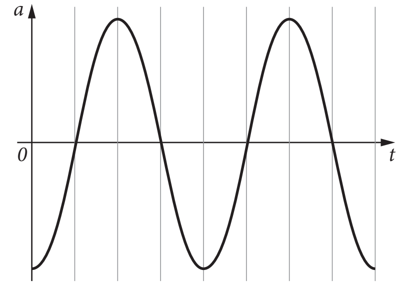
$x = 3\cos 2x$



$v = \dot{x} = -6\sin 2x$



$a = \ddot{x} = -12\cos 2x = -4x$



By looking at a vertical line through all three graphs you can see that:

- when $t = 0$, initially, x is a maximum, $v = 0$ and A is a minimum
- when x is first zero, then v is a minimum and A is zero
- when x is a minimum, $v = 0$ and A is a maximum
- when x is next zero, v is a maximum and A is zero
- then the process repeats itself.

Another way of looking at these links is:

- When the particle passes through the origin (or centre of motion) the speed is greatest and the acceleration is zero.
- When the particle is at its greatest distance from the centre of motion, the velocity is zero (particle at rest) and the acceleration takes on its greatest or least value (being the opposite sign of the displacement).

Thus, given the graph of either the displacement, velocity or acceleration, it is possible to determine information about the unknown parts of the motion.

Example 23

A particle moves in simple harmonic motion with $\dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right)$.

(a) Sketch a velocity–time graph for this particle for $0 \le t \le 2\pi$.

(b) If the particle is initially at the origin, find the equation for the displacement of the particle.

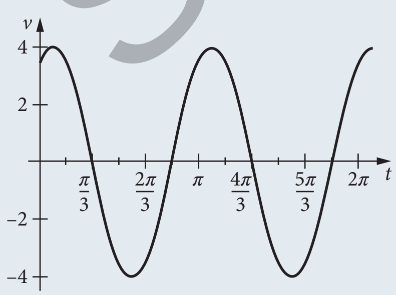
(c) Sketch the displacement–time graph for this particle for $0 \le t \le 2\pi$.

(d) Find the equation for the particle's acceleration.

(e) Describe the motion of the particle for the first 2π seconds.

Solution

(a) $\dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right)$



(b) $t = 0,\ x = 0$: $\quad x = \int 4\cos\left(2t - \frac{\pi}{6}\right)dt$

$$x = 4 \times \frac{1}{2}\sin\left(2t - \frac{\pi}{6}\right) + C$$

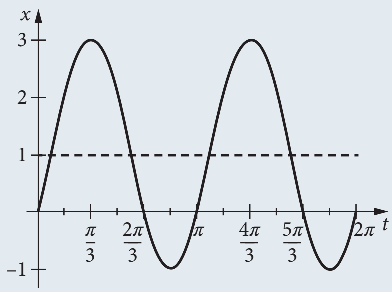
$$x = 2\sin\left(2t - \frac{\pi}{6}\right) + C$$

$t = 0, x = 0$: $\quad 0 = 2\sin\left(-\frac{\pi}{6}\right) + C$

$$C = 1$$

$$x = 2\sin\left(2t - \frac{\pi}{6}\right) + 1$$

(c) $x = 2\sin\left(2t - \frac{\pi}{6}\right) + 1$



(d) $\dot{x} = 4\cos\left(2t - \frac{\pi}{6}\right)$

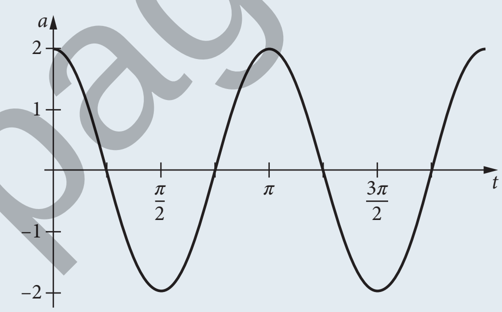
$\ddot{x} = 4 \times (-2)\sin\left(2t - \frac{\pi}{6}\right)$

$\ddot{x} = -8\sin\left(2t - \frac{\pi}{6}\right) = -4x + 4$

(e) The particle exhibits simple harmonic motion, oscillating around $x = 1$ with a maximum displacement of 2 units, i.e. it oscillates between $x = -1$ and $x = 3$. It starts at $x = 0$ and finishes after 2π seconds at $x = 0$. The amplitude of the motion is 2 units, the period is π seconds.

Example 24

The graph shows the acceleration of a particle undergoing simple harmonic motion.



(a) Find the equation for the particle's acceleration.

(b) If the particle is initially at rest at the origin, find the equation of the particles velocity.

(c) Find the equation for the displacement and state the amplitude and period of the particle's motion.

(d) Sketch the displacement–time graph for this particle.

Solution

(a) From the graph, $\ddot{x} = 2$ when $t = 0$ and this is the maximum value.

Let the equation be $\ddot{x} = b\cos(nt)$

$t = 0, \ddot{x} = 2$: $\quad 2 = b\cos 0$, so $b = 2$

The period of the graph is π, so $\pi = \frac{2\pi}{n}$, so $n = 2$.

Hence $\ddot{x} = 2\cos(2t)$.

(b) $t = 0, \dot{x} = 0, x = 0$: $\quad \dot{x} = \int 2\cos(2t)\,dt$

$\dot{x} = \sin(2t) + C$

$t = 0, \dot{x} = 0$: $\quad 0 = \sin 0 + C$, so $C = 0$

$\dot{x} = \sin(2t)$

(c) $t = 0, x = 0$: $\quad x = \int \sin(2t)\,dt$

$x = -\frac{1}{2}\cos(2t) + C$

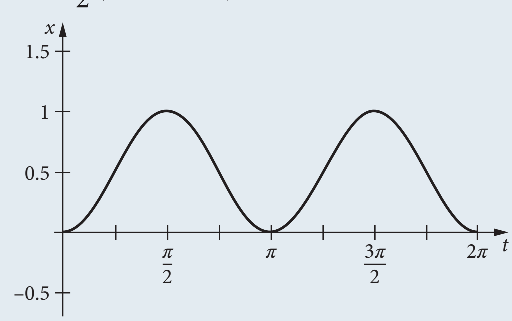
$t = 0, x = 0$: $\quad 0 = -\frac{1}{2}\cos 0 + C$

$C = \frac{1}{2}$

$x = -\frac{1}{2}\cos(2t) + \frac{1}{2} = \frac{1}{2}(1 - \cos(2t))$

Amplitude $= \frac{1}{2}$, Period $= \pi$

(d) $x = \frac{1}{2}(1 - \cos(2t))$



Example 25

A particle is moving in a straight line with SHM. The velocity of the particle is respectively $\sqrt{5}$ m s^{-1} and 2 m s^{-1} at distances of 1 m and 2 m from the centre of motion. Find:

(a) the length of the path **(b)** the period of the motion.

Solution

(a) Given $x = 1$, $v = \sqrt{5}$; $x = 2$, $v = 2$.

Hence use: $v^2 = n^2(A^2 - x^2)$

At $x = 1, v = \sqrt{5}$: $5 = n^2(A^2 - 1)$ [1]

At $x = 2, v = 2$: $4 = n^2(A^2 - 4)$ [2]

[1] ÷ [2]: $\dfrac{5}{4} = \dfrac{n^2(A^2 - 1)}{n^2(A^2 - 4)}$

$$5A^2 - 20 = 4A^2 - 4$$
$$A^2 = 16$$
$$A = 4 \quad \text{as } A > 0$$

The amplitude of the motion is 4 m, so the length of the path ($2A$) is 8 m.

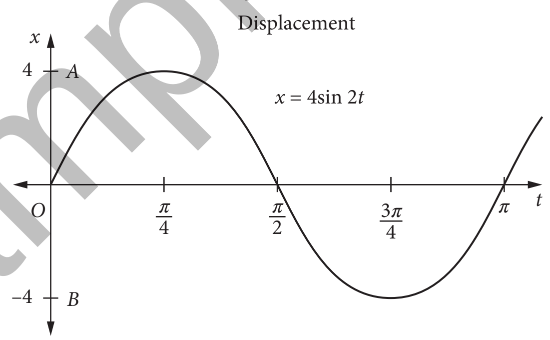
(b) Substitute $A = 4$ into [1]: $5 = 15n^2$

$$\therefore \quad n = \frac{1}{\sqrt{3}} \quad \text{as } n > 0$$

Thus the period of the motion is $T = \dfrac{2\pi}{n} = 2\pi\sqrt{3}$ seconds.

Simple harmonic motion about the point $x = c$

In Example **16**, you considered a particle moving in a straight line with displacement x m from a fixed point O at time t seconds defined by the equation $x = 4\sin 2t$. The graph of the displacement is given in the following diagram.

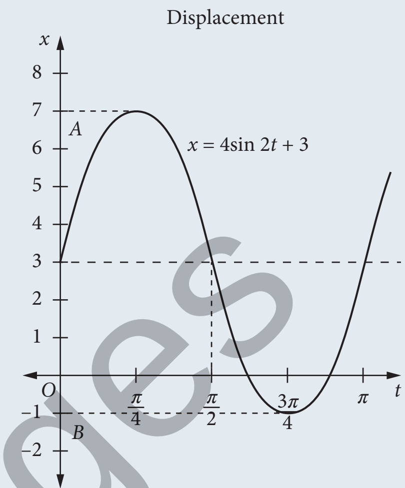


Simple harmonic motion about the point $x = c$ will involve a vertical translation of the trigonometric graph. This will be considered in the following example.

Example 26

The equation of motion changes to $x = 4\sin 2t + 3$. By considering the graph of this function, analyse the motion of the particle, and find expressions for **(a)** velocity, **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution



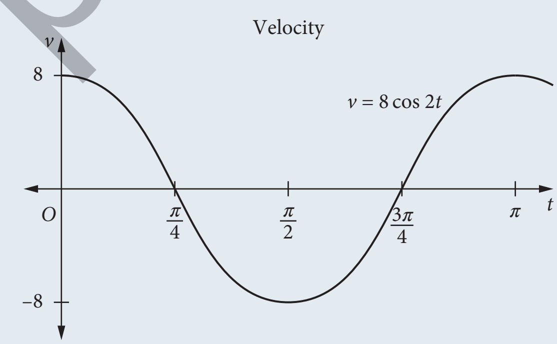
The effect of the +3 on the equation $x = 4\sin 2t$ is to translate the graph 3 units upwards with no horizontal translation.

The graph shows the following points.

- When $t = 0$, $x = 3$.
- When $t = \frac{\pi}{4}$, $x = 7$.
- When $t = \frac{\pi}{2}$, $x = 3$.
- When $t = \frac{3\pi}{4}$, $x = -1$.
- When $t = \pi$, $x = 3$.

The particle is moving in a straight line along the x-axis. It starts at $x = 3$ and takes $\frac{\pi}{4}$ seconds to move to A, a distance of 4 m. It takes another $\frac{\pi}{4}$ seconds to return to $x = 3$ and then another $\frac{\pi}{4}$ seconds to move to B, a point 4 m below $x = 3$. It then takes $\frac{\pi}{4}$ seconds to return to $x = 3$.

The pattern repeats every π seconds and the particle oscillates about $x = 3$, the centre of the motion, with a period of π. The amplitude of the motion is 4.



(a) $x = 4\sin 2t + 3$

$$\frac{dx}{dt} = 8\cos 2t$$

This is the same as the velocity function in Example **14**, so shifting the centre of the motion makes no change to the velocity of the motion as a function of time.

The particle is instantaneously at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 3$, $v = 8$ so the particle is travelling towards A with its greatest velocity.

When $t = \frac{\pi}{2}$, $x = 3$, $v = -8$ so the particle is travelling towards B with its least velocity.

Squaring the velocity function gives $v^2 = 64\cos^2 2t$

$$v^2 = 64\left(1 - \sin^2 2t\right)$$

But $\sin 2t = \frac{x-3}{4}$: $v^2 = 64\left(1 - \left(\frac{x-3}{4}\right)^2\right)$

$$v^2 = 4\left(16 - (x-3)^2\right)$$

This gives the velocity in terms of x and reflects the shift of 3 units in the centre of the motion.

(b) $\dot{x} = v = 8\cos 2t$

$$\ddot{x} = \frac{dv}{dt} = -16\sin 2t$$

This is the same as the acceleration function in Example **14**, so shifting the centre of the motion makes no change to the acceleration of the motion as a function of time.

But $\sin 2t = \frac{x-3}{4}$: $\ddot{x} = -16 \times \frac{x-3}{4}$

$$\ddot{x} = -4(x-3)$$

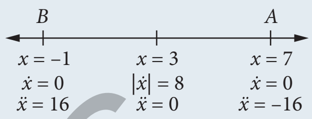
This gives the acceleration in terms of x and reflects the shift of 3 units in the centre of the motion. It shows that the acceleration is always directed towards the centre of motion.

Thus when $x = 3$, $\ddot{x} = 0$: the acceleration is zero when the particle passes through the centre of the motion.

When $x = 7$, $\ddot{x} = -16$: the acceleration is least when the displacement is greatest.

When $x = -1$, $\ddot{x} = 16$: the acceleration is greatest when the displacement is least.

(c) In this motion, the particle moves in a straight line so that its acceleration is always directed towards the fixed point in the line that is the centre of the motion. The magnitude of this acceleration is proportional to its distance from the centre of the motion.



$x = 3$ is the centre of the motion, when the particle is at A $(x > 3)$ its acceleration is towards $x = 3$ $(\ddot{x} < 0)$, while when the particle is at B $(x < 3)$ its acceleration is again towards $x = 3$ $(\ddot{x} > 0)$.

Thus the equations of motion for simple harmonic motion about the point $x = c$ may be written as:

Displacement:	$x = A\sin(nt+\alpha) + c$	$x = A\cos(nt+\alpha) + c$
Velocity:	$\dot{x} = An\cos(nt+\alpha)$	$\dot{x} = -An\sin(nt+\alpha)$
Acceleration:	$\ddot{x} = -An^2\sin(nt+\alpha)$	$\ddot{x} = -An^2\cos(nt+\alpha)$
or:	$\ddot{x} = -n^2(x-c)$	

where $x = c$ is the centre of the motion, $(x - c)$ the displacement of the particle from the centre of motion at c, A is the amplitude, $\frac{2\pi}{n}$ is the period and $\frac{\alpha}{n}$ the phase shift.

Example 27

A particle is moving with SHM about the point $x = 5$ cm. The particle starts from rest at the point $x = 14$ cm with a period of 4π seconds. Calculate:

(a) the amplitude **(b)** the acceleration when $t = 4$.

Solution

(a) The centre of motion is $x = 5$ so the displacement is given by: $x = A\cos(nt + \alpha) + 5$

$t = 0$, $x = 14$: $14 = A\cos\alpha + 5$

$$A\cos\alpha = 9$$

Velocity is given by: $\dot{x} = -An\sin(nt+\alpha)$

$t = 0$, $\dot{x} = 0$: $0 = -An\sin\alpha$

$$\alpha = 0$$
$$A\cos 0 = 9$$
$$A = 9$$

The amplitude is 9.

(b) Period: $4\pi = \frac{2\pi}{n}$

$$n = \frac{1}{2}$$

$$\dot{x} = -9n\sin nt$$

$n = \frac{1}{2}$: $\dot{x} = -\frac{9}{2}\sin\frac{t}{2}$

$$\ddot{x} = -\frac{9}{4}\cos\frac{t}{2}$$

$t = 4$: $\ddot{x} = -\frac{9}{4}\cos 2$

$$\approx 0.936\,\text{cm s}^{-2}$$

It is also possible to start with the acceleration given as $\ddot{x} = -n^2(x - c)$, and given the initial conditions, to find the equations for the velocity, x, and displacement, v, in terms of t.

Example 28

A particle is moving in a straight line under simple harmonic motion. Its acceleration is given by $\ddot{x} = -4(x-1)$. It starts from rest at a point 2 metres to the right of the origin with a velocity $2\sqrt{3}\text{ m s}^{-1}$.
Find the equation for the velocity and displacement as functions of time.

Solution

$\ddot{x} = -4(x-1)$

$n^2 = 4$, so $n = 2$.

The centre of motion is $x = 1$.

Let $x = 1 + A\cos(2t - \alpha)$

$\dot{x} = -2A\sin(2t - \alpha)$

$t = 0, x = 2: \quad 2 = 1 + A\cos(-\alpha)$

$$1 = A\cos(-\alpha)$$
$$A\cos\alpha = 1$$

$t = 0, \dot{x} = 2\sqrt{3}: \quad 2\sqrt{3} = -2A\sin(-\alpha)$

$$-\sqrt{3} = A\sin(-\alpha)$$
$$A\sin\alpha = \sqrt{3}$$
$$\frac{A\sin\alpha}{A\cos\alpha} = \frac{\sqrt{3}}{1}$$
$$\tan\alpha = \sqrt{3}$$
$$\alpha = \frac{\pi}{3}$$

$$A\cos\frac{\pi}{3} = 1$$
$$\frac{A}{2} = 1$$
$$A = 2$$

Hence the equation for the displacement is $x = 1 + 2\cos\left(2t - \frac{\pi}{3}\right)$ and the equation for the velocity is $\dot{x} = -4\sin\left(2t - \frac{\pi}{3}\right)$.

Example 29

Assume that the tides of the ocean rise and fall in SHM. The depth of water y metres in a harbour channel is given by $y = 10 - 2\cos\frac{\pi t}{6}$ where t is the number of hours after low tide. On a particular day, low tide occurs at 7 a.m. when the channel is 8 metres deep. On the same day, high tide occurs at 1 p.m. when the channel is 12 metres deep.

(a) Show that these values satisfy the equation $y = 10 - 2\cos\frac{\pi t}{6}$.

(b) What is the depth of the channel at 10 a.m.?

(c) State the amplitude and period of the motion.

(d) Draw the graph of $y = 10 - 2\cos\frac{\pi t}{6}$ for $0 \le t \le 12$.

(e) A ship needs at least 9 metres of water in the channel to be able to pass through safely. Use your graph to find when the ship can safely pass through the channel.

(f) Find the answer to part **(e)** algebraically.

Solution

(a) At 7 a.m., $t = 0$:

$y = 10 - 2\cos(0) = 10 - 2 = 8\,\text{m}$

The depth at 7 a.m. is 8 metres.

At 1 p.m., $t = 6$:

$y = 10 - 2\cos(\pi) = 10 - 2\times(-1) = 10 + 2 = 12\,\text{m}$

The depth at 1 p.m. is 12 metres.

(b) 10 a.m., $t = 3$:

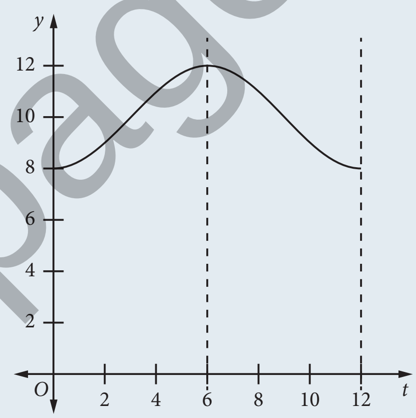
$y = 10 - 2\cos\left(\frac{\pi}{2}\right) = 10 - 0 = 10\,\text{m}$

The depth at 10 a.m. is 10 metres.

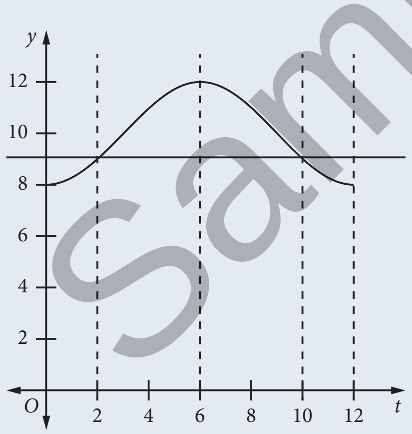
(This is halfway between low tide and high tide.)

(c) Amplitude = 2 m; period = 12 hours

(d)



(e) Draw the line $y = 9$ on the graph. When the curve is above the line it is safe to pass through the channel.



It is safe to pass through the channel from 2 hours after low tide (i.e. after 9 a.m.) until 10 hours after low tide (i.e. before 5 p.m.).

(f) To find the answer algebraically, solve $10 - 2\cos\left(\frac{\pi t}{6}\right) \ge 9$ as an equation and interpret the result:

$$2\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\frac{\pi t}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\therefore \quad t = 2, 10$$

In summary, the solution to the differential equation for simple harmonic motion, $\ddot{x} = -n^2 x$, can take one of three forms, i.e. $x = A\sin(nt + \alpha)$, $x = A\cos(nt + \alpha)$ or $x = A\sin(nt + \alpha) + B\cos(nt + \beta)$.

The first two results have already been shown to be solutions.

Verify that $x = C\sin(nt) + D\cos(nt) + c$ is a solution to the differential equation.

$\dot{x} = nC\cos(nt) - nD\sin(nt)$

$$\begin{aligned}\ddot{x} &= -n^2C\sin(nt) - n^2D\cos(nt)\\ &= -n^2\left(C\sin(nt) + D\cos(nt)\right)\\ &= -n^2(x-c)\end{aligned}$$

This is the differential equation for simple harmonic motion about $x = c$.

Simple harmonic motion about $x = c$

We have shown that the differential equation of motion about the point $x = c$ is $\ddot{x} = -n^2(x-c)$.

The corresponding displacement and velocity equations are: $x = A\cos(nt+\alpha) + c$
$\dot{x} = -An\sin(nt+\alpha)$

Consider
$$\begin{aligned}v &= -An\sin(nt+\alpha)\\ v^2 &= A^2n^2\sin^2(nt+\alpha)\\ v^2 &= A^2n^2(1-\cos^2(nt+\alpha))\\ v^2 &= n^2(A^2 - A^2\cos^2(nt+\alpha))\\ v^2 &= n^2\left[\left(A^2-(x-c)^2\right)\right]\end{aligned}$$
where A is the amplitude and c the centre of the motion.

Example 30

The speed, v cm s^{-1}, of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where the magnitude of its displacement from a fixed point, O, is x cm.

Show that the motion is simple harmonic and find:

(a) the centre of the motion
(b) the period of the motion
(c) the amplitude of the motion.

Solution

Method 1

$$v^2 = 2\left(3+2x-x^2\right)$$
$$v^2 = 2\left[4-\left(1-2x+x^2\right)\right]$$
$$v^2 = 2\left[4-(x-1)^2\right]$$

Hence the motion is simple harmonic.

Method 2

$$v^2 = 2\left[3+2x-x^2\right]$$
$$\frac{1}{2}v^2 = 3+2x-x^2$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2-2x$$
$$\ddot{x} = -2(x-1)$$

Hence the motion is simple harmonic.

(a) Centre of motion is $x = 1$.
(b) $n^2 = 2$ so $n = \sqrt{2}$.
Period $= \dfrac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$
(c) $A^2 = 4$ so amplitude = 2 cm.

EXERCISE 6.3 SIMPLE HARMONIC MOTION (SHM)

1 The displacement x m of a particle moving in a straight line is given by $x = 6\cos 4t$. Describe the motion of the particle.

2 The equation of motion of a particle moving with simple harmonic motion is $\ddot{x} = -9x$. Find its period, amplitude and greatest speed if: **(a)** $x = 0, \dot{x} = 2$ when $t = 0$ **(b)** $x = 2, \dot{x} = 2$ when $t = 0$.

3 A particle is moving in a straight line. If x metres is its displacement at time t seconds and $\left(\frac{dx}{dt}\right)^2 = 5\left(4 - x^2\right)$, find the acceleration in terms of x only. Show that the motion is simple harmonic and find its period and amplitude.

4 The velocity v metres per second of a particle moving in simple harmonic motion along the x-axis is given by the equation $v^2 = 100 - 4x^2$. The amplitude in metres of the motion of the particle is:

A 2 **B** 5 **C** 10 **D** 50

5 The displacement of a particle at time t is given by $x = 8\sin 4t + 15\cos 4t$. What is the maximum velocity of the particle?

A $2\sqrt{17}$ **B** 17 **C** 34 **D** 68

6 A particle undergoing simple harmonic motion in a straight line has an acceleration given by $\ddot{x} = 36 - 6x$, where x is the displacement after t seconds. Where is the centre of the motion?

A $x = 0$ **B** $x = 6$ **C** $x = 12$ **D** $x = 18$

7 The displacement of a particle at time t is given by $x = 5\sin 2t + 12\cos 2t$. What is the maximum acceleration of the particle?

A 52 **B** 26 **C** 13 **D** 4

8 A particle is moving along the x-axis in simple harmonic motion centred at the origin.
When $x = 2$, the velocity of the particle is 5.
When $x = 5$, the velocity of the particle is 4. Find:

(a) the amplitude of the motion **(b)** the period of the motion.

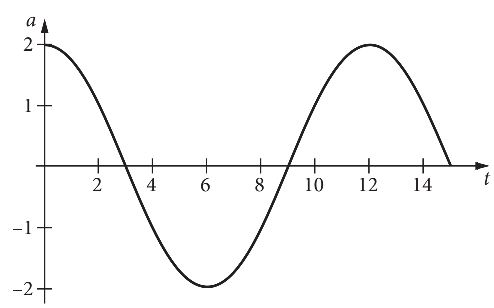
9 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$. Express the acceleration in terms of x only and hence show that the motion is simple harmonic. Find:

(a) the period **(b)** the amplitude **(c)** the speed when $x = -2.5$ **(d)** the acceleration when $x = -2.5$.
(e) Sketch the displacement–time graph for this particle for $0 \le t \le 6.3$.
(f) Sketch a velocity–time graph for this particle for $0 \le t \le 6.3$.
(g) Describe the motion of the particle for the first 6 seconds.

10 A particle moves in simple harmonic motion with $\dot{x} = 4\sin\left(2t - \frac{\pi}{6}\right)$.

(a) Sketch a velocity–time graph for this particle for $0 \le t \le 2\pi$.
(b) If the particle is initially at the origin, find the equation for the displacement of the particle.
(c) Sketch the displacement–time graph for this particle for $0 \le t \le 2\pi$.
(d) Find the equation for the particle's acceleration.
(e) Describe the motion of the particle for the first 2π seconds.

11 The graph shows the acceleration of a particle undergoing simple harmonic motion.



(a) Find the equation for the particle's acceleration.
(b) If the particle is initially at rest at the origin, find the equation of the particle's velocity.
(c) Find the equation for the displacement and state the amplitude and period of the particle's motion.
(d) Sketch the displacement–time graph for this particle.

12 The speed of a point moving along the x-axis is $v\ \text{m s}^{-1}$ and the displacement of the point from the origin is x m. Prove in each case below that the motion is simple harmonic, finding **(i)** the centre of motion and **(ii)** the period and amplitude, when v is given by:

(a) $v^2 = 300 + 100x - 25x^2$ **(b)** $v^2 = 128 - 32x - 16x^2$ **(c)** $v^2 = 6 + 4x - 2x^2$

13 A particle is moving in SHM along the x-axis. Its velocity v at position x is given by $v^2 = 30 - 4x - 2x^2$. Find:

(a) all values of x for which the particle is at rest
(b) an expression for the acceleration of the particle in terms of x
(c) the maximum speed of the particle.

14 The displacement x metres at time t seconds of a point moving in a straight line is given by $x = a\cos(nt + \varepsilon)$. Find the form that this expression takes if initially:

(a) $\dot{x} = 0$ and $x = -5$ **(b)** $x = 0$ and the velocity is negative.

15 The velocity of a particle moving in a straight line is $3\sqrt{16 - x^2}\ \text{m s}^{-1}$ where x m is the displacement of the particle from a fixed point in the line. Is the motion simple harmonic? Find the acceleration when $x = 4$. What is the maximum speed?

16 Solve the differential equation $\frac{d^2x}{dt^2} + 16x = 0$ subject to the conditions $x = 3$ and $\frac{dx}{dt} = 16$ when $t = 0$. Find the maximum displacement and the maximum speed if x metres is the displacement of the particle moving in a straight line at time t seconds.

17 A particle moves with simple harmonic motion. If the particle starts from the equilibrium position (centre of motion) with velocity $4\ \text{m s}^{-1}$ towards the origin and the period is $\frac{\pi}{2}$ seconds, find:

(a) the displacement at time t **(b)** the amplitude.

18 A particle moves with simple harmonic motion. When it is 2 m from its equilibrium position, its velocity is $6\ \text{m s}^{-1}$; when it is 3 m from equilibrium, its velocity is $4\ \text{m s}^{-1}$. Find the period of this motion and the amplitude.

19 If $x = a\sin nt + b\cos nt$, find the velocity and acceleration of a particle whose displacement from a fixed point O is x at time t. Show that $\ddot{x} = -n^2x$ and find the amplitude and maximum speed.

20 The position x m of a particle relative to a fixed point O at any time t seconds is $x = 5 - 2\cos^2 t$. By finding the acceleration in terms of x, show that the motion is simple harmonic. Find:

(a) the centre of motion **(b)** the period **(c)** the amplitude.
(d) Sketch the displacement–time graph for this particle for $0 \le t \le 2\pi$.
(e) Sketch a velocity–time graph for this particle for $0 \le t \le 2\pi$.
(f) Describe the motion of the particle for the first 2π seconds.

21 A particle executes simple harmonic motion of period 8 seconds and amplitude 10 m. Calculate the velocity and acceleration when its displacement is 6 m from the centre of motion. Find also its maximum acceleration.

22 A particle moves in a straight line so that its position x metres from a fixed point O at time t seconds is given by $x = 10 + 8\sin 2t + 6\cos 2t$. Prove that the motion is simple harmonic. Find the period and amplitude.

23 A particle moves in a straight line. At time t seconds its distance from a point O on the line is x metres. The following is an incomplete table of observations:

t	0	7	9	11	18
x	0			0.5	0

Complete the table using two different assumptions, namely:

(a) that the particle moves with uniform acceleration
(b) that the particle performs simple harmonic motion with a period of 12 seconds.

24 A floating buoy oscillates up and down with the waves, rising and falling 2 metres about its mean position. Find its greatest velocity and acceleration if the period of the motion is 3 seconds.

25 A particle moving with SHM starts from rest at $x = 5$ and after two seconds reaches $x = 2.5$. Find:

(a) an expression for the displacement at time $t \ge 0$ **(b)** the speed at $x = 0$
(c) the amplitude, frequency and period **(d)** the maximum speed **(e)** the maximum acceleration.

26 The amplitude of a particle moving with SHM is 5 m and the acceleration when 2 m from the mean position is 4 m s^{-2}. Find the speed of the particle when at the mean position and when 4 m from the mean position.

27 A particle is moving with SHM of period π seconds and a maximum velocity of 8 metres per second. Find the amplitude and the velocity at a distance of 3 metres from the central position.

28 A particle executes SHM of period $\frac{2\pi}{\sqrt{5}}$ seconds. If it starts from rest with displacement 10 metres, find:

(a) the frequency **(b)** the amplitude.

29 A point moving with SHM has a speed of 5 m s^{-1} when passing through its mean position O. Find the speed and acceleration when it is 1.5 m from O, given that the period is π seconds.

30 A particle is moving with SHM of amplitude 10 metres. Find how much time it takes to travel 6 metres from its mean position if the period is 10 seconds.

31 A point moves with SHM in such a way that its speed is 8 and 6 m s^{-1} respectively at distances 3 and 4 m from the mean position. Calculate the period of the motion and the magnitude of the greatest acceleration.

32 Assume that the tides rise and fall in SHM. At low tide the channel in a harbour is 9 m deep and at high tide it is 12 m deep. Low tide is at 9 a.m. and high tide is at 4 p.m.

(a) What is the depth of the channel at 12:30 p.m.? **(b)** What is the amplitude of the motion?
(c) What is the period of the motion?
(d) Show that the depth y m of the water in the channel is given by $y = 10.5 - 1.5\cos\frac{\pi t}{7}$ where t is the number of hours after low tide.
(e) A ship needs at least 10 m of water to pass through the channel safely. Between what times can the ship safely navigate the channel?

33 Assume that over several days of constant weather, the cycle of changing temperature each day is simple harmonic between 13°C at 4 a.m. and 23°C at 4 p.m.

(a) What is the temperature at 10 a.m.?
(b) Show that the pattern of temperature can be represented by $T = 18 - 5\cos\frac{\pi t}{12}$ where T is the temperature in °C and t is the time in hours after 4 a.m.
(c) At what time of day would the temperature be: **(i)** 18°C **(ii)** 15°C **(iii)** 21°C?

34 A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 3 m s^{-1} and its maximum acceleration is 8 m s^{-2}. Find the amplitude and period of the motion.

35 **(a)** A particle is travelling in a straight line. Its displacement from the origin is x metres at time t seconds. If $x = \cos 2t - \sqrt{3}\sin 2t$, express x in the form $R\cos(2t + \alpha)$, where $R > 0$ and $0 \le \alpha < 2\pi$.
(b) Find the maximum speed of the particle and the time at which it first occurs.

36 A particle moves in a straight line and its position at time t is given by $x = 4 + \sqrt{3}\sin 3t - \cos 3t$.

(a) If $\sqrt{3}\sin 3t - \cos 3t = R\sin(3t - \alpha)$, where α is in radians, find the value of R and α.
(b) Prove that the particle is undergoing simple harmonic motion.
(c) Find the amplitude and the centre of the motion.
(d) Find when the particle is first at its minimum displacement.

37 A particle with displacement x and velocity v is moving in simple harmonic motion with acceleration, $\ddot{x}$, given by $\ddot{x} = -12x$. The particle is initially at rest at $x = -4$.

(a) What is the period of the motion? **(b)** Show that $v^2 = 12(16 - x^2)$. **(c)** Find x as a function of time.

38 A particle moves along a straight line and its displacement, x centimetres, from a fixed point O at a given time t seconds is given by $x = 4 + \cos^2 t$.

(a) Show that its acceleration is given by $\ddot{x} = 18 - 4x$.
(b) Explain why the motion is simple harmonic.
(c) Find the centre, amplitude and period of the motion.

39 A particle is moving in a straight line under simple harmonic motion. It has a displacement of x metres from a point O, on the line, at time t seconds given by $x = 1 + 2\cos\left(2t - \frac{\pi}{4}\right)$.

(a) Show that $\ddot{x} = -4(x-1)$.

(b) Find the centre of the motion and the time taken for the particle to first reach maximum speed.

(c) Find the amplitude of the motion and when the particle is first at rest.

40 The tide can be modelled using simple harmonic motion. At a particular location, the depth at high tide is 5 metres and the depth at low tide is 1 metre. At this location, the tide completes two full periods every 25 hours. Let x represent the depth in metres and t be the time in hours after the first low tide of the day.

(a) If this depth of this tide can be modelled by the function $x = a\cos nt + c$, find the values of a, n and c. The first low tide today is at 2 a.m.

(b) At what time is the first high tide today?

(c) At what time this evening is the depth of water increasing at the fastest rate?

41 The deck of a ship is 2.4 m below the level of a wharf at low tide and 0.6 m above wharf level at high tide. Low tide is at 8:30 a.m. and high tide is at 2:35 p.m. Find the time when the deck is level with the wharf, assuming the motion of the tides is simple harmonic.

42 A particle moves in a straight line with simple harmonic motion. Its speed at distances x_1, x_2 from the centre of its motion are v_1, v_2 respectively. Show that:

(a) the period of the motion is $2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

(b) the amplitude is $\sqrt{\dfrac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$

43 A particle moves in a straight line under a force such that it describes simple harmonic motion of amplitude a about a point O. The time of a complete oscillation is $5T$. The particle is released from rest at a point A, where $OA = a$. At time T, another particle describing an exactly similar motion is released from rest at A.

(a) Show that the two particles first meet at time $2T$ after the release of the second particle.

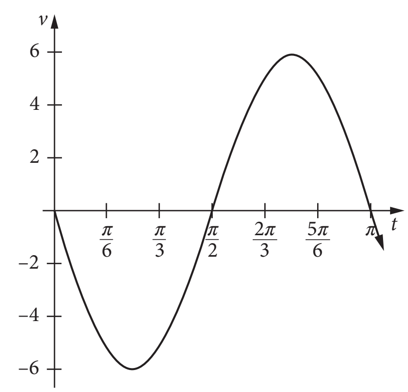
(b) Find the distance from O to the point where the particles first meet.

44 A particle is moving under simple harmonic motion. The graph of its velocity is shown in the diagram.

(a) Find the equation of the velocity.

(b) What is the amplitude and period of the particle's motion?

(c) If the particle starts from rest at the origin, find the equation for the displacement of the particle.

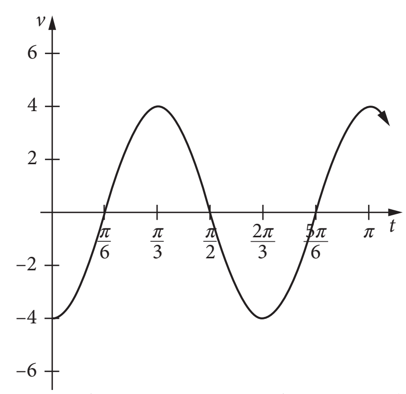


45 A particle is moving under simple harmonic motion. The graph of its displacement is shown in the diagram.

(a) Find the equation of the displacement.

(b) What is the amplitude and period of the particle's motion?

(c) Find initial velocity and acceleration of the particle.



46 A particle is moving in a straight line under simple harmonic motion. Its acceleration is given by $\ddot{x} = -4(x-2)$. It starts from rest at a point 3.5 metres to the right of the origin with a velocity $-3\sqrt{3}\,\text{m s}^{-1}$. Find the equation for the velocity and acceleration as functions of time.

6.4 OTHER EXAMPLES OF MOTION

Displacement, velocity, acceleration—important connections

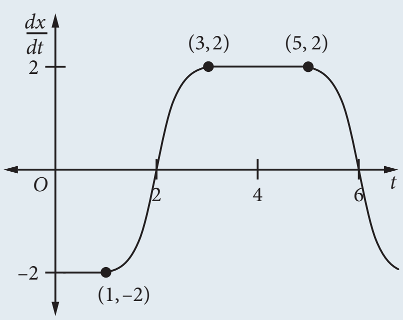
Given the displacement function, you can find the velocity and acceleration functions by differentiating with respect to time.

Given the velocity function, you can find the displacement function by integrating with respect to time, and you can find the acceleration function by differentiating with respect to time.

If the information is given as a graph, then you must remember that the definite integral can be seen as the area under the graph, while the derivative can be seen as the gradient of the curve. If you have the graph of the velocity $\frac{dx}{dt}$ against time, then the value of the definite integral gives the displacement and the slope of the curve gives the acceleration.

Example 31

The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.



(a) At what times is the velocity zero?

(b) At what time is the displacement x from the origin a maximum?

(c) What is the displacement when $t = 4$? What does this tell you?

(d) When does the particle have zero acceleration?

(e) At what time is the acceleration the greatest?

(f) Use the trapezoidal rule to estimate the displacement when $t = 6$.

Solution

(a) $\frac{dx}{dt} = 0$: graph shows this at $t = 2, 6$.

Hence the velocity is zero (particle at rest) at 2 seconds and again at 6 seconds.

(b) Maximum and minimum displacement occur when $\frac{dx}{dt} = 0$.

From the graph, the value of $x = \int_0^2 \frac{dx}{dt}\,dt < 0$ so the displacement at $t = 2$ is a minimum.

The value of $x = \int_0^6 \frac{dx}{dt}\,dt > 0$ so the displacement at $t = 6$ is a maximum.

(c) From the symmetry of the graph, $x = \int_0^4 \frac{dx}{dt}\,dt = 0$, so the displacement is zero and the particle is again at the origin at $t = 4$.

(d) The particle has zero acceleration when the velocity is constant, i.e. when the graph is horizontal. Acceleration is zero for $0 \le t \le 1$, $3 \le t \le 5$.

(e) The velocity graph is steepest at $t = 2$ and $t = 6$. At $t = 2$, the slope > 0; at $t = 6$, the slope < 0. Greatest acceleration is at $t = 2$.

(f) As the particle is back at the origin when $t = 4$, the displacement when $t = 6$ is given by $x = \int_4^6 \frac{dx}{dt}\,dt$.

From the graph, you have the points $(4, 2)$, $(5, 2)$, $(6, 0)$: $x = \int_4^6 \frac{dx}{dt}\,dt \approx \frac{1}{2}\left(2 + 2 \times 2 + 0\right) = 3$.

The particle is about 3 units on the positive side of the origin.

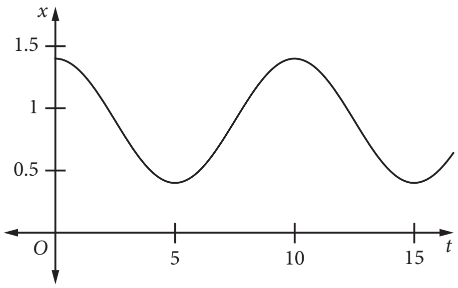
EXERCISE 6.4 OTHER EXAMPLES OF MOTION

1 The displacement x of a particle at time t is given by $x = 6\cos 4t + 3$. Find:

(a) the velocity and acceleration at any time t **(b)** the position of the particle when $t = 0$

(c) the values that x can take **(d)** the time when the particle first reaches the position $x = 0$.

2 Consider the graph.



Which of the following functions does this graph represent?

A $x = 0.9 + \cos\dfrac{\pi t}{10}$ **B** $x = 0.9 + \cos\dfrac{\pi t}{5}$

C $x = 0.9 + 0.5\cos\dfrac{\pi t}{10}$ **D** $x = 0.9 + 0.5\cos\dfrac{\pi t}{5}$

3 The position x metres of a particle relative to a fixed point O at any time t is $x = 4 - \cos 2t$.

(a) Sketch the graph of x as a function of t in the domain $0 \le t \le 2\pi$.

(b) Find the times when the particle is at rest.

(c) Express the acceleration in terms of: **(i)** t **(ii)** x

4 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is $2\sin t$, and $v = 1$ and $x = 1$ when $t = 0$, find x as a function of t.

5 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\sin\dfrac{\pi t}{2}$, $0 \le t \le 4$.

(a) Sketch the graph of its displacement at any time t. **(b)** At what times is the particle at rest?

(c) Find the speed when $t = 2.5$. **(d)** Express the acceleration in terms of: **(i)** t **(ii)** x

6 A particle moves in a straight line so that at time t its displacement from a fixed origin O is x, where $x = 2 + t - 2\cos t$.

(a) Write the velocity and acceleration at any time t.

(b) Find its initial displacement, velocity and acceleration.

7 A particle moves in a straight line so that its displacement x from a fixed origin at any time t is given by $x(t) = 2(1 - e^{-t})$.

(a) Find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$. **(b)** Sketch the graph of $x(t)$. **(c)** Find t when $x(t) = 1$.

8 A particle moves in a straight line. At time t its displacement from a fixed origin on the line is x, where $x = 2 - 2\sin 2t$, $0 \le t \le 2\pi$.

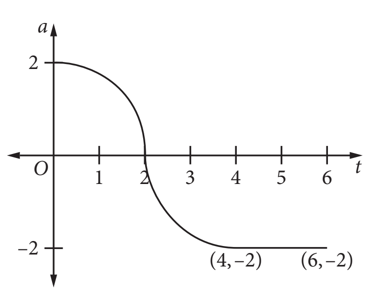
(a) Draw the graph of x as a function of t.

(b) Show that the particle oscillates between $x = 0$ and $x = 4$.

(c) For what values of t is the velocity zero?

(d) Express the acceleration in terms of: **(i)** t **(ii)** x

9 A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration a of the particle as a function of time t for $0 \le t \le 6$.

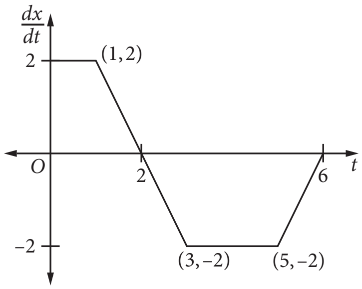


(a) Write the time at which the velocity of the particle is a maximum.

(b) At what time during the interval $0 < t \le 6$ is the particle at rest?

(c) At what time during the interval $0 \le t \le 6$ is the particle farthest from the origin? Give brief reasons for your answer.

10 The graph shows the velocity $\frac{dx}{dt}$ of a particle as a function of time. Initially the particle is at the origin.



(a) At what time is the displacement x from the origin a maximum?

(b) At what time does the particle return to the origin? Justify your answer.

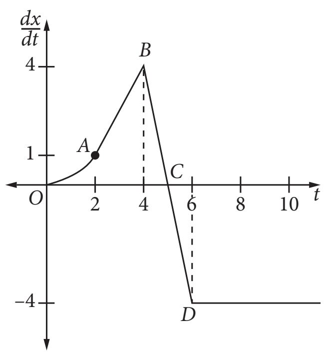
(c) Draw a sketch of the acceleration $\frac{d^2x}{dt^2}$ as a function of time for $0 \le t \le 6$.

11 The table shows the velocity (in metres per second) of a moving object, evaluated at one-second intervals.

t	0	1	2	3	4	5	6
v	0	4.6	5.7	8.0	9.9	12.7	18.2

Use the trapezoidal rule to estimate the distance travelled over the time interval $0 \le t \le 6$.

12 An object is moving on the x-axis. The graph shows the velocity $\frac{dx}{dt}$ of the object as a function of time t. The coordinates of the points shown on the graph are $A(2, 1)$, $B(4, 4)$, $C(5, 0)$ and $D(6, -4)$. The velocity is constant for $t \ge 6$.



(a) Using the trapezoidal rule, estimate the distance travelled between $t = 0$ and $t = 4$.

(b) How far is it from B to C?

(c) The object is initially at the origin. During which time(s) is the displacement of the object decreasing?

(d) Estimate the time at which the object returns to the origin. Justify your answer.

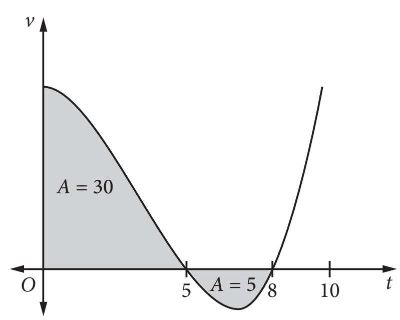
(e) Sketch the displacement x as a function of time.

13 The velocity–time graph at right shows the first 10 seconds of motion of an object moving in a straight line. The graph also shows the areas between the curve and the t-axis from $t = 0$ to $t = 5$ and from $t = 5$ to $t = 8$.

(a) Find the distance travelled by the object before it first comes to rest.

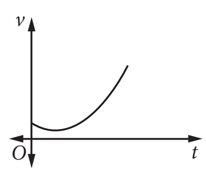
(b) Find the total distance travelled by the object before it comes to rest for the second time.

(c) If the initial displacement is 20 metres, find the displacement when the object comes to rest for the second time.

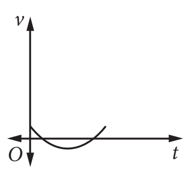


14 Each graph below shows the velocity–time relationship for an object moving in a straight line. In which case does the object change its direction only once?

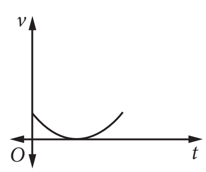
A



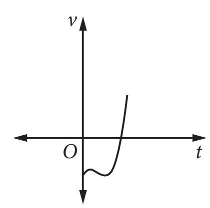
B



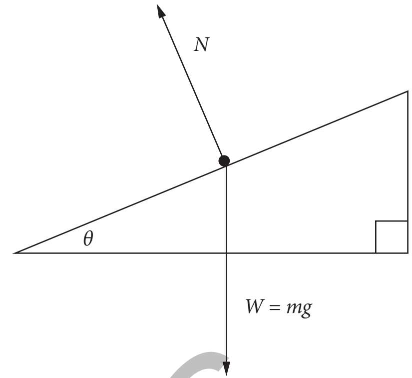
C



D



6.5 MODELLING MOTION WITHOUT RESISTANCE—INCLINED PLANES



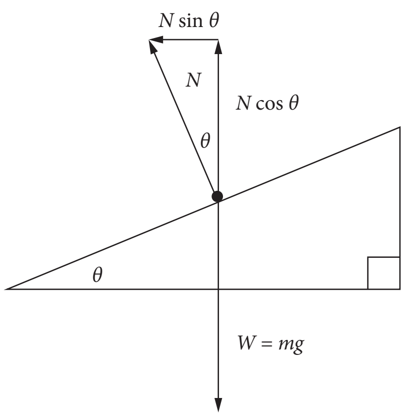
Forces are vector quantities that can be resolved into mutually perpendicular components. These components may be parallel to and perpendicular to an inclined plane, or parallel to and perpendicular to the surface of the earth (ground).

The resulting forces in each direction, according to Newton's first and second laws, should equal the object's mass multiplied by its acceleration in that direction.

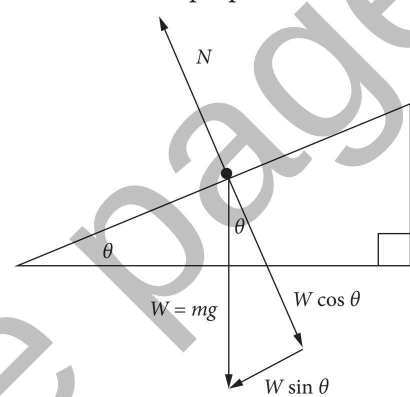
In the diagram, the particle is resting a smooth inclined plane inclined at an angle θ to the horizontal.
The forces acting on the particle are the normal force perpendicular to the plane and the particle's weight acting downwards.

These forces may be resolved in two different ways as shown in the following diagrams.

Vertical and horizontal



Parallel to and perpendicular to the plane

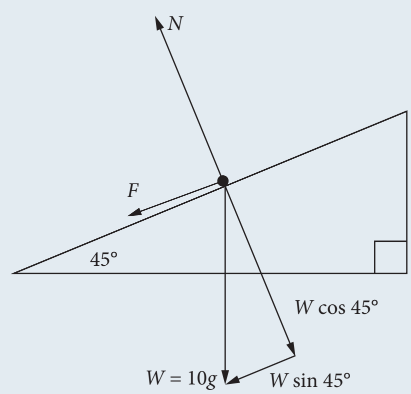


Use $g = 9.8\ \text{m s}^{-2}$, unless given otherwise.

Example 32

A mass of 10 kg moves from rest on a smooth plane inclined at 45° to the horizontal. Find the normal force and the equation of the velocity, v, in terms of time, t.

Solution



Resolve parallel and perpendicular to the plane.

Normal to plane: $N = 10g\cos 45°$

$N = 69.3$ newtons

Parallel to plane: $F = 10g\sin 45°$

$10\ddot{x} = 10g\sin 45°$

$$\ddot{x} = \frac{g}{\sqrt{2}}$$

$$v = \int \frac{g}{\sqrt{2}}\,dt$$

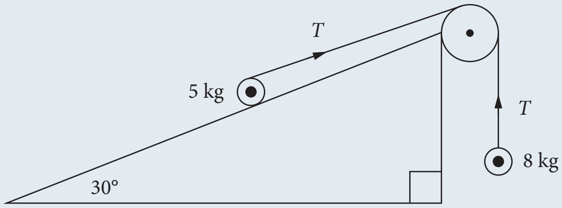
$$v = \frac{\sqrt{2}}{2}gt + C$$

$t = 0,\ v = 0$: $\quad C = 0$

$$v = \frac{\sqrt{2}}{2}gt$$

Example 33

A mass of 5 kg sits on a smooth plane which is inclined at 30° to the horizontal. The mass is connected to a light inextensible string to a mass of 8 kg via a frictionless pulley.



The 5 kg mass is released from rest and begins to slide up the plane.

(a) Find an expression for the velocity, v, of the 5 kg mass in terms of time, t, as it moves up the plane.

(b) Find the tension in the string.

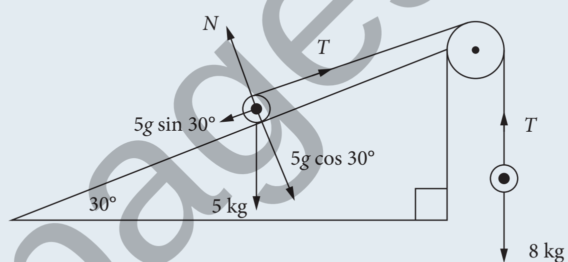
Solution

(a) Resolve forces along the plane.

For 5 kg particle: $N = 5g\cos 30°$

$T - 5g\sin 30° = 5\ddot{x}$

$T = 5\ddot{x} + \frac{5}{2}g$



For 8 kg particle: $8g - T = 8\ddot{x}$

Substitute: $8g - \left(5\ddot{x} + \frac{5}{2}g\right) = 8\ddot{x}$

$13\ddot{x} = \frac{11}{2}g$

$\ddot{x} = \frac{11}{26}g$

$\dot{x} = \int \frac{11g}{26}\,dt$

$\dot{x} = \frac{11g}{26}t + C$

$t = 0, v = 0$: $C = 0$

$v = \frac{11g}{26}t$

(b) $\ddot{x} = \frac{2T - 5g}{10}$

and $\ddot{x} = \frac{8g - T}{8}$

$\frac{2T - 5g}{10} = \frac{8g - T}{8}$

$8T - 20g = 40g - 5T$

$13T = 60g$

$T = \frac{60 \times 9.8}{13} = 45.23\,\text{N}$

EXERCISE 6.5 MODELLING MOTION WITHOUT RESISTANCE – INCLINED PLANES

1 A mass slides down a frictionless plane inclined at an angle of 30° to the horizontal. What is the acceleration down the plane?

2 A 10 kg block is placed on a smooth inclined plane, inclined at 25° to the horizontal.

(a) What are the components of the weight parallel to and perpendicular to the plane?
(b) If the block is released from rest and slides down the plane, what is its velocity after 1 second?

3 A block of mass *m* kg starts from rest and slides 2 metres down a frictionless plane inclined at an angle or 30°.

(a) How long does it take to reach the bottom?
(b) What is its velocity as it reaches the bottom?

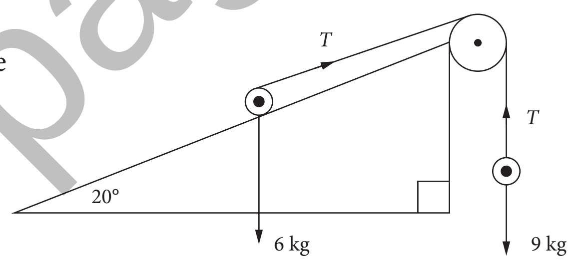
4 Two smooth inclined planes have the same height, but different slopes, with the angle of inclination being 30° and 60°.

If the same mass is released at the top of each inclined plane, which mass reaches the bottom first?

5 A 5 kg mass is held stationary on a frictionless inclined plane, angle of inclination 30°, by a **horizontal** force, F. Find F.

6 A 4 kg mass is pulled up a smooth inclined plane with a force $F = 30$ N parallel to the plane. If the angle of inclination of the plane is 20°, find the acceleration of the mass.

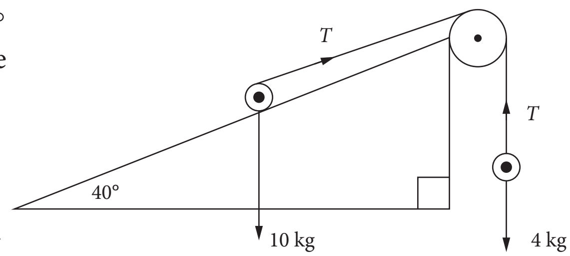
7 A mass of 6 kg sits on a smooth plane which is inclined at 20° to the horizontal. The mass is connected by a light inextensible string to a mass of 9 kg via a frictionless pulley.



The 6 kg mass is released from rest and begins to slide up the plane.

(a) Find an expression for the velocity, v, of the 6 kg mass in terms of time, t, as it moves up the plane.
(b) Find the tension in the string.

8 A mass of 10 kg sits on a smooth plane which is inclined at 40° to the horizontal. The mass is connected by a light inextensible string to a mass of 4 kg via a frictionless pulley.



The 10 kg mass is released from rest and begins to slide down the plane.

(a) Find an expression for the velocity, v, of the 10 kg mass in terms of time, t, as it moves down the plane.
(b) Find the tension in the string.

9 A particle of mass *m* kg is on a smooth inclined plane that makes and angle θ with the horizontal. The particle is attached to a light inextensible string that passes over a smooth pulley at the top of the plane. The other end of the string is connected to a hanging particle of mass *M* kg. The system is released from rest and moves without friction.

Assume the string remains taut and the acceleration of both masses is a, and gravity is g.

Which of the following pairs of equations correctly represents the motion of the two masses?

A $mg\sin\theta - T = ma$
$Mg - T = Ma$

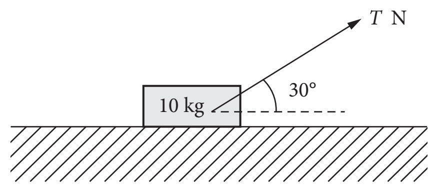
B $mg\cos\theta - T = ma$
$Mg - T = -Ma$

C $T - mg\cos\theta = ma$
$Mg - T = Ma$

D $T - mg\sin\theta = ma$
$Mg - T = Ma$

10 A 10 kg box sits on frictionless ground and is subjected to a force of T newtons inclined 30° up from the horizontal.

If the box starts from rest and moves to the right, find its velocity after t seconds in terms of T.



11 A variable force acts on a particle, causing it to move in a straight line. At time t seconds, where $t \geq 0$, its velocity v metres per second and position x metres from the origin are such that $v = e^x \sin x$.

The acceleration of the particle in m s^{-2}, can be expressed as:

A $e^x(\sin x + \cos x)$

B $e^x \sin x(\sin x + \cos x)$

C $\frac{1}{2}e^{2x}\sin^2 x$

D $e^{2x}\left(\sin^2 x + \frac{1}{2}\sin 2x\right)$

12 A particle is moving in a straight line with acceleration a m s^{-2} given by $a = v^2 - 3$. The velocity of the particle, at displacement x metres from the origin is v m s^{-1}.

Find v in terms of x, given that $v = -2$ where $x = 1$.

13 A particle of mass 5 kg moves in a straight line subject to a force F.

The particle was initially at rest at the origin. If $F = 15 - 10e^{-t}$, determine the displacement of the particle at time t seconds.

14 A particle which was initially at rest, subsequently moves in a straight line along the x-axis.

Its acceleration is given by $\ddot{x} = 2v + 1$ m s^{-2}, where v m s^{-1} is the velocity of the particle after t seconds.

By expressing $\ddot{x}$ as $\frac{dv}{dt}$, find an expression for $v(t)$.

6.6 MOTION WITH CONSTANT ACCELERATION

Consider motion of a particle where the acceleration is constant. Let the particle start from the origin with an initial velocity of u ms^{-1}. (Also called uniform acceleration).

We have $\frac{dv}{dt} = a$, where a is a constant.

Hence $v = \int a\,dt$.

So, $v = at + c$

$t = 0, v = u : u = c$

Hence $v^2 = \frac{dx}{dt} = u + at$ [1]

$x = \int (u + at)dt$

$x = ut + \frac{1}{2}at^2 + C$

$t = 0, x = 0 : C = 0$

$x = ut + \frac{1}{2}at^2$

For uniform acceleration, the displacement or distance covered is often denoted by s, so the formula becomes

$s = ut + \frac{1}{2}at^2$ [2]

Eliminating t from the two equations gives another useful result for uniform acceleration.

[1] becomes: $t = \frac{v - u}{a}$ Substitute in (2): $s = \frac{u(v - u)}{a} + \frac{a(v - u)^2}{2a^2}$

$2as = 2uv - 2u^2 + (v^2 - 2uv + u^2)$

$2as = v^2 - u^2$

$v^2 = u^2 + 2as$

Example 34

A particle moves in a straight line with constant acceleration. It starts moving to the right with velocity $15\,\text{m}\,\text{s}^{-1}$. After 3 seconds its velocity is $9\,\text{m}\,\text{s}^{-1}$. Find:

(a) How far the particle travels before it comes to rest;

(b) how long it takes until the particle returns to its starting point;

(c) the times when the particle is 50 m to the right of its starting point, and its velocity at those times.

Solution

(a) $u=15\cdot t=3, v=9:\quad v=u+at$

$$9=15+3a$$

$$3a=-6$$

$$a=-2\,\text{m}\,\text{s}^{-2}$$ (The velocity reduced so the acceleration is negative.)

(b) $u=15\cdot s=0, a=-2:\quad s=ut+\frac{1}{2}at^2$

$$0=15t+0.5\times(-2)\times t^2$$

$$t^2-15t=0$$

$$t(t-15)=0$$

$$t=0,15$$

The particle returns to the starting point after 15 seconds.

(c) $u=15\cdot s=50, a=-2:\quad s=ut+\frac{1}{2}at^2$

$$50=15t-t^2$$

$$t^2-15t+50=0$$

$$(t-5)(t-10)=0$$

$$t=5,10$$

$t=5:\qquad v=15+(-2)\times 5=15-10=5\,\text{m}\,\text{s}^{-1}$

$t=10:\qquad v=15+(-2)\times 10=15-20=-5\,\text{m}\,\text{s}^{-1}$

The particle is 50 m from the starting point after 5 seconds and has a velocity of $5\,\text{m}\,\text{s}^{-1}$, and again after 10 seconds when it has a velocity of $-5\,\text{m}\,\text{s}^{-1}$.

Motion under gravity

When you neglect air resistance, motion under gravity has a constant acceleration of $g\,\text{m}\,\text{s}^{-2}$, where $g=9.8$. Occasionally, for ease of calculation, it may be given that $g=10$.

Example 35

A rocket is projected vertically upwards from the ground. By the time its fuel. has burnt out, it has reached a speed of $56\,\text{m}\,\text{s}^{-1}$ and a height of 40 metres. From this point onwards it is subject only to acceleration due to gravity. Find the height to which it rises.

Solution

Taking a point 40 m above the ground as the starting point and the positive direction as upwards, the $a = -9.8$.

Reaches greatest height when $v = 0$.

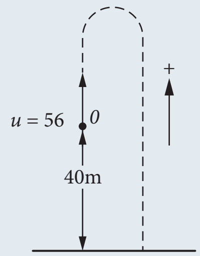
$t = 0, u = 56, v = 0, a = -9.8: \; v^2 = u^2 + 2as$

$$0 = 56^2 + 2 \times (-9.8) \times s$$

$$19.6s = 3136$$

$$s = 160$$

Height reached $= 40 + 160 = 200$ m.



Example 36

A stone is thrown vertically upwards from the top of a cliff 30 m high, with a speed of $14\,\text{m s}^{-1}$. Find the speed with which it hits the ground at the bottom of the cliff and the time it takes to reach the ground.

Solution

(a) Take the origin as the top of the cliff and the positive direction as upwards.

Thus $u = 14,\ a = -9.8,\ s = -30: \; v^2 = u^2 + 2as$

$$v^2 = 14^2 + 2 \times (-9.8)(-30)$$

$$v^2 = 196 + 588$$

$$v^2 = 784$$

$$v = \pm\sqrt{784} = \pm 28$$

Since the stone is moving in the negative direction when it hits the ground, its velocity is negative, so $v = -28\ \text{m s}^{-1}$.

It strikes the ground with a speed $28\ \text{m s}^{-1}$.

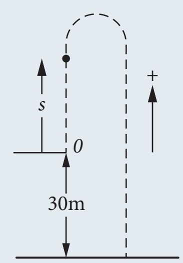
(b) For time of flight, use $v = u + at$

$v = -28,\ u = 14,\ a = -9.8: \quad -28 = 14 - 9.8t$

$$9.8t = 42$$

$$t = \frac{42}{9.8} \approx 4.3\,\text{sec}$$

It takes about 4.3 seconds for the stone to reach the ground.



EXERCISE 6.6

1 A drag racer can cover the quarter mile (403.2 m) from rest in 6.94 seconds. Find the acceleration, assuming it to be constant, and express it as a multiple of g. Also find the racer's speed at the end of the quarter mile.

2 **(a)** **(i)** What constant retardation must be given to a car travelling at 50 km/h (13.89 m/s) to bring it to rest in a distance of 50 m?

(ii) How long will it take to stop?

(iii) If the initial speed is 100 km/h, what will be the distance covered in coming to rest using the same retardation?

(b) **(i)** What constant retardation must be given to a car travelling at V m/s to bring it to rest in a distance of d m?

(ii) How long will it take to stop?

(iii) Show that doubling the speed increases the braking distance fourfold if the retardation is the same.

3 A train is travelling at 25 m/s when the brakes are applied, reducing the speed to 10 m/s in 10 seconds. Assuming constant acceleration, find:

(a) the acceleration of the train whilst braking
(b) the distance travelled in the first 10 seconds after the brakes are applied
(c) how long it will take to stop the train from its speed of 25 m/s
(d) how far the train will travel before stopping.

4 **(a)** A train travels a distance of 2000 metres in 100 seconds whilst accelerating uniformly from rest. What is its velocity at the end of the first 1000 metres?

(b) A particle moves a distance d in time t whilst accelerating uniformly from rest. What is its velocity after travelling half the distance?

5 **(a)** A bus moving with uniform acceleration has a speed of 4 m/s at A and 12 m/s at B. What is its speed midway between A and B? How does this speed compare with its average speed from A to B?

(b) A particle moving with uniform acceleration has speed U at A and speed V at B. What is its speed midway between A and B? How does this speed compare with its average speed from A to B?

6 **(a)** If a tram moving with constant acceleration takes 20 seconds to travel 150 m from A to B, find its speed after 10 seconds. How does this speed compare with its average speed from A to B?

(b) If it takes time T for a particle moving with constant acceleration to travel a distance d from A to B, find its speed at time $\frac{1}{2}T$. How does this speed compare with its average speed?

7 **(a)** A stone dropped from the top of a tower reaches the ground in 3.5 seconds. How high is the tower, and with what speed does the stone strike the ground?

(b) A particle falls freely from a height of 40 metres. Find its speed on reaching the ground and the time taken for the fall.

8 A particle is projected vertically upwards from a point 70 m above the ground with a speed of 35 m s^{-1}.

(a) How long will it take to reach the ground?
(b) What is its speed then?

9 An object is projected vertically upwards from the top of a building and reaches the ground after 3 seconds at a speed three times the speed of projection. Find the speed of projection and the height of the building.

10 A particle is projected vertically upwards from a point O on the ground with a speed of 14 m s^{-1}.

(a) Find the maximum height reached.
(b) Find the velocity and height above O at time t.
(c) How long does it take for the particle to return to O?

6.7 RECTILINEAR RESISTED MOTION

Whenever a body moves through a medium (such as air, water, oil etc.), it is subjected to a resistance that acts in the opposite direction to the motion. If the body falls vertically downwards, the resistance acts upwards; if the body is projected vertically upwards, the resistance acts downwards. (Of course the gravitational weight force mg always acts vertically downwards.)

In general, the faster the body moves, the greater the resistance. Air resistance is typically proportional to some power of the speed, so that air resistance = kv^n. This topic will consider this for $n = 1$ and $n = 2$, i.e. air resistance proportional to the speed of the body ($n = 1$) and air resistance proportional to the square of the speed of the body ($n = 2$). These are the two most useful models for air resistance.

It is important to realise that the effect of air resistance (or other resistance) is always one of retardation:

- **Resistance** is force acting in the opposite direction to the motion.
- **Resistive Force** (measured in newtons) is the force acting on an object in the opposite direction to the motion.
- **Retardation** is acceleration acting in the opposite direction to the motion.

In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.

Resisted motion in a horizontal line

Example 37

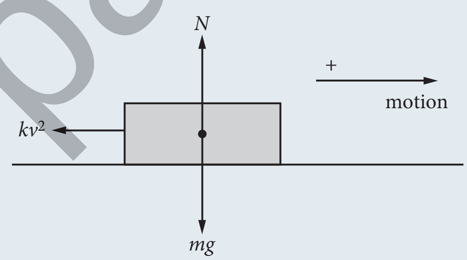
A particle of mass m and initial speed v_0 moves on a horizontal surface against a resistance proportional to the square of the speed. Express the velocity in terms of the distance travelled.

Solution

Take the origin O as the particle's initial position. Take the particle's direction of motion as the positive direction.

Vertically, there is no motion (i.e. the particle moves always at the same horizontal level), so there is a zero resultant force acting in the vertical direction: $\therefore\ N - mg = 0$

Horizontally, there is motion. The only force acting horizontally is the resistance:



$$m\ddot{x} = -kv^2$$

$$\ddot{x} = -\frac{k}{m}v^2$$

$$v\frac{dv}{dx} = -\frac{k}{m}v^2$$

$$\frac{dv}{dx} = -\frac{k}{m}v$$

$$\therefore \frac{dx}{dv} = -\frac{m}{kv}, \quad v \neq 0$$

$$\int_0^x dx = -\frac{m}{k}\int_{v_0}^{v}\frac{dv}{v} \qquad \text{(from start } x = 0,\ v = v_0 \text{ to end } x = x,\ v = v\text{)}$$

$$x = -\frac{m}{k}\left[\log_e v\right]_{v_0}^{v}$$

$$x = -\frac{m}{k}\log_e\frac{v}{v_0}$$

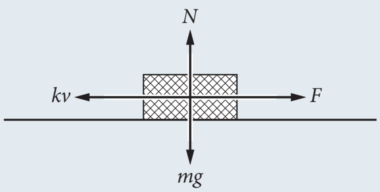
$$v = v_0 e^{-\frac{kx}{m}} \qquad \text{(making } v \text{ the subject)}$$

Example 38

A child pulls a sled across ice by exerting a for of F newtons. The mass of the sled is m kg, the friction between the sled and the ice is negligible, but the sled is subject to air resistance of magnitude k times the speed. If the sled starts from rest, find expressions for:

(a) velocity as a function of time
(b) displacement as a function of time
(c) displacement as a function of velocity
(d) the terminal velocity.

Solution



$$F - kv = m\ddot{x}$$

$$\ddot{x} = \frac{F - kv}{m}$$

(a)

$$\frac{dv}{dt} = \frac{F - kv}{m}$$

$$\frac{dt}{dv} = \frac{m}{F - kv}, v \neq \frac{F}{k}$$

$$t = \int \frac{m}{F - kv} dv$$

$$t = -\frac{m}{k} \ln|F - kv| + C$$

$$t = 0, v = 0 : 0 = -\frac{m}{k} \ln F + C$$

$$C = \frac{m}{k} \ln F$$

$$t = -\frac{m}{k} \ln|F - kv| + \frac{m}{k} \ln F$$

$$t = \frac{m}{k} \ln\left|\frac{F}{F - kv}\right|$$

$$\frac{kt}{m} = \ln\left|\frac{F}{F - kv}\right|$$

$$e^{\frac{kt}{m}} = \frac{F}{F - kv}$$

$$F - kv = Fe^{-\frac{kt}{m}}$$

$$kv = F - Fe^{-\frac{kt}{m}}$$

$$v = \frac{F}{k}\left(1 - e^{-\frac{kt}{m}}\right)$$

(b)

$$\frac{dx}{dt} = \frac{F}{k}\left(1 - e^{-\frac{kt}{m}}\right)$$

$$x = \frac{F}{k} \int \left(1 - e^{-\frac{kt}{m}}\right) dt$$

$$= \frac{F}{k}\left(t + \frac{m}{k} e^{-\frac{kt}{m}}\right) + C$$

$$t = 0, x = 0 : 0 = \frac{F}{k}\left(0 + \frac{m}{k}\right) + C$$

$$C = -\frac{Fm}{k^2}$$

$$x = \frac{F}{k}\left(t + \frac{m}{k} e^{-\frac{kt}{m}}\right) - \frac{Fm}{k^2}$$

$$x = \frac{F}{m}\left(t + \frac{m}{k} e^{-\frac{kt}{m}} - \frac{m}{k}\right)$$

(c) Use $v\frac{dv}{dx}=\frac{F-kv}{m}$

$$\frac{dx}{dv}=\frac{mv}{F-kv},\ F\neq kv$$

$$x=\int\frac{mv}{F-kv}\,dv$$

$$=\frac{m}{k}\int\frac{kv}{F-kv}\,dv$$

$$=-\frac{m}{k}\int\frac{F-kv+F}{F-kv}\,dv$$

$$=-\frac{m}{k}\int\left(1-\frac{F}{F-kv}\right)dv$$

$$=-\frac{m}{k}\left(v+\frac{F}{k}\ln\left|F-kv\right|\right)+C$$

$$x=0,\ v=0:0=-\frac{m}{k}\left(0+\frac{F}{k}\ln F\right)+C$$

$$C=\frac{mF}{k^2}\ln F$$

$$x=-\frac{m}{k}\left(v+\frac{F}{k}\ln\left|F-kv\right|\right)+\frac{mF}{k^2}\ln F$$

$$x=\frac{m}{k}\left(\frac{F}{k}\ln F-v-\frac{F}{k}\ln\left|F-kv\right|\right)$$

$$x=\frac{m}{k}\left(\frac{F}{k}\ln\left|\frac{F}{F-kv}\right|-v\right),\ 0\leq v\leq\frac{F}{k}$$

(d) From (a), $v=\frac{F}{k}\left(1-e^{-\frac{kt}{m}}\right)$.

For limiting velocity, $t\to\infty:\lim\limits_{t\to\infty}v=\frac{F}{k}(1-0)=\frac{F}{k}$.

EXERCISE 6.7 RECTILINEAR RESISTED MOTION

1 A particle of unit mass moves horizontally under a resisting force that is proportional to the cube of the speed. Find:

(a) the speed of the particle when it has travelled a distance d after the instant when its speed is u

(b) the time taken to travel a distance d after the instant when its speed is u.

2 A particle has an initial velocity U. After travelling a distance d in time T along a straight horizontal path, its velocity is V. The resistive force on the particle at any time is proportional to its velocity at that time.

Show that: **(a)** $V=U-kd$ **(b)** $U=Ve^{kT}$ **(c)** $U=Ve^{\frac{T(U-V)}{d}}$

3 As a truck moves along a horizontal straight road it experiences a horizontal driving force of T newtons (from the engine) and a resistance force. The resistance force is mkv^2 newtons, where m kg is the mass of the truck, v m s^{-1} is its speed at time t seconds and k is a positive constant. The equation of motion of the truck is:

A $m\frac{dv}{dt}=T-kv^2$ **B** $m\frac{dv}{dt}=T-mkv^2$ **C** $\frac{dv}{dx}=T-kv$ **D** $m\frac{dv}{dx}=T-mkv^2$

4 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mkv+mkv^2$ newtons where its velocity is v m s^{-1} and k is a positive constant. At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity v m s^{-1}. Which of the following is an expression for x in terms of v?

A $\frac{1}{k}\int\frac{1}{1+v}\,dv$ **B** $\frac{1}{k}\int\frac{1}{v+v^2}\,dv$ **C** $-\frac{1}{k}\int\frac{1}{v+v^2}\,dv$ **D** $-\frac{1}{k}\int\frac{1}{1+v}\,dv$

5 The acceleration of a particle P moving in a straight line is given by $a=-x^{-2}$, $x \neq 0$, where x is the displacement from the origin.

The particle is initially at rest at a point P, which is at a distance of 6 metres to the right of O.

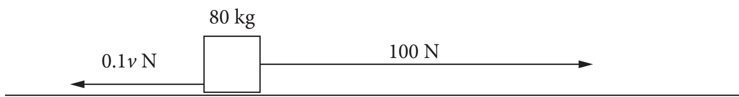
The particle then moves in a straight line towards O.

(a) Show that $\dfrac{dx}{dt}=-\sqrt{2}\sqrt{\dfrac{6-x}{x}}$.

(b) Using the substitution $x=6\cos^2\theta$, show that the time taken to reach a position 3 metres to the right of O can be given by $t=12\sqrt{3}\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta$.

(c) Hence find t, the time taken to reach a distance 3 metres to the right of O, in exact form.

6 An 80 kg mass is being dragged along a horizontal surface by a force of 100 newtons.



The mass is subject to friction of $0.1v$ newtons, where v is the speed of the mass in metres per second.

The mass is initially at rest.

The equation of motion is $\ddot{x}=\dfrac{1000-v}{800}$.

(a) Show that the equation for time as a function of velocity is $t=800\ln\left(\dfrac{1000}{1000-v}\right)$.

(b) Find the velocity of the mass after 10 seconds, correct to one decimal place.

7 A car with mass of 1000 kg starts from rest at a traffic light and drives on a horizontal freeway. The car's engine provides a total force of 16 000 newtons and the combined resistance from friction from the freeway's surface and air resistance is $10v^2$ newtons, where v is the velocity at time t seconds.

(a) Show that the car's acceleration is given by $\ddot{x}=16-0.01v^2$, and find the speed the car cannot exceed on the freeway (i.e the limiting speed).

(b) Show that $x=50\ln\left(\dfrac{1600}{1600-v^2}\right)$ and find the distance the car has travelled when it achieves a speed of $30\,\text{m s}^{-1}$.

8 A swimmer in a pool stops swimming and is slowed with a resistance force $R=-mk\left(v_0+v^2\right)$, where m is the mass of the swimmer, v_0 is the velocity of the swimmer when she stops swimming, x is the distance travelled, t is the time and v is the velocity of the swimmer after she stops swimming.

(a) Show that the distance before the swimmer comes to rest is $x=\dfrac{1}{2k}\ln\left(1+v_0\right)$.

(b) Show that the time t after she stops swimming is given by $t=\dfrac{1}{k\sqrt{v_0}}\tan^{-1}\left(\dfrac{v_0-v}{(1+v)\sqrt{v_0}}\right)$, and hence find an expression for the time when the swimmer comes to rest.

9 A train of mass m, pulled by a locomotive which exerts a constant (propelling) force P, is moving at speed $v\,\text{m s}^{-1}$ along a straight level track against a resistive force mkv, where k is a positive constant. The speed increases from $2\,\text{m s}^{-1}$ to $4\,\text{m s}^{-1}$ over a time interval of 5 seconds.

(a) Show that $P=2km\left(\dfrac{2e^{5k}-1}{e^{5k}-1}\right)$.

(b) Find the corresponding distance moved.

(c) There is an upper limit to the speed that the train can attain. Find the value of the upper limit.

10 A particle of mass 1 kg moves in a straight line such that at time t seconds its displacement from a fixed origin is x metres and its speed is $v\,\text{ms}^{-1}$. If the resultant force, in newtons, is:

(a) $3+2t$, find its displacement in terms of t, given that $v=1$ and $x=2$ when $t=0$

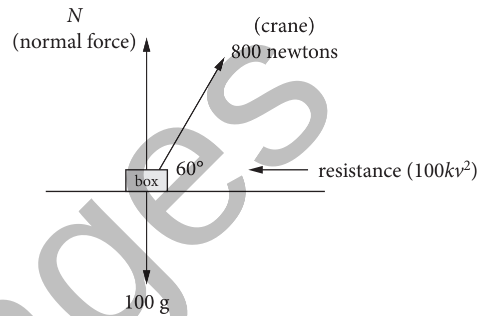
(b) $3-2x$, find v in terms of x, given that $v=2$ when $x=1$

(c) $4-v^2$, find t in terms of v, given that $v=0$ when $t=0$.

11 A particle of mass m, initially with speed v_0, moves horizontally against a resistance proportional to the square of the speed. Express the velocity in terms of the distance travelled.

12 A particle of mass m moves under a retardation force that is proportional to the cube of the speed. Find how long it takes to travel a distance d from the instant when its speed was u.

13 A box with mass of 100 kg is attached to a crane by a taut rope which is at an angle of 60° to the horizontal. The box is initially stationary, but it then starts to move horizontally along the ground as the crane pulls it with and overall force of 800 newtons.



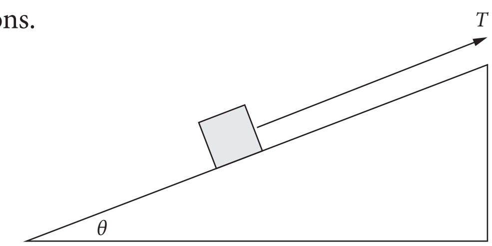
Once the box starts moving it experiences resistance in the form of friction of $100\,kv^2$, where v is the velocity, as shown in the diagram.

(a) Taking $g = 10\,\text{m s}^{-2}$, by resolving the forces vertically and assuming the box stays on the ground, find the magnitude of the normal force.

(b) By resolving forces horizontally, show that $\ddot{x} = 4 - kv^2$, and find the value of k if the box has a limiting horizontal velocity of $20\,\text{m s}^{-1}$.

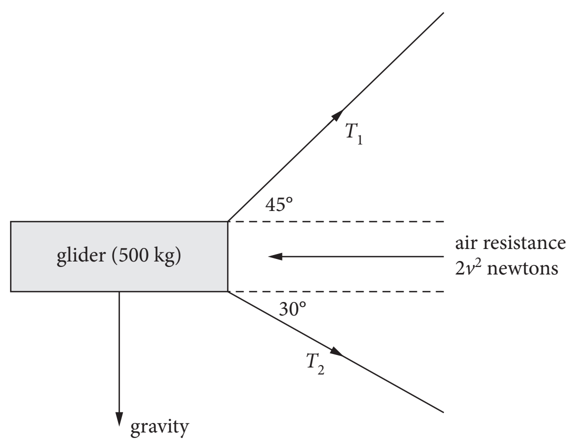
(c) Show that $x = -\dfrac{1}{2k}\ln\left(1-\dfrac{kv^2}{4}\right)$, and find how far the box has moved when it is moving at $10\,\text{m s}^{-1}$.

14 A 2 kg mass is being pulled up a slope by a string of tension 10 newtons. The slope is at an angle θ to the horizontal and the coefficient of friction is 0.3, meaning the force due to friction is $0.3N$. As well as friction, the force gravity g, and the normal force N, are also acting on the mass.



By resolving forces with forces along and perpendicular to the slope, find the net force F_{net}, in newtons, up the slope.

15 A 500 kg glider is being pulled behind an aeroplane which is flying horizontally. It is attached to the aeroplane by two ropes at angles of 45° above and 30° below the horizontal and the tensions in the two ropes are labelled T_1 and T_2 respectively. The glider experiences air resistance of $2v^2$ newtons. The accceleration due to gravity is $10\,\text{m s}^{-2}$. (see diagram).

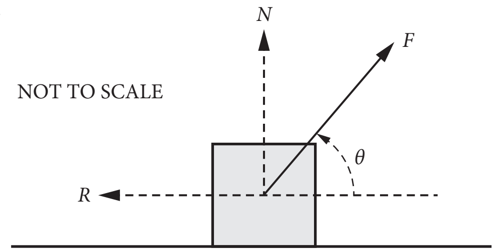


(a) By resolving forces vertically, show that $T_2 = T_1\sqrt{2} - 10000$.

(b) Given $T_1 = 10\,000$ newtons, by resolving forces horizontally, find the total resultant force on the glider. Hence show that $\ddot{x} = 10\left(\sqrt{2}+\sqrt{6}-\sqrt{3}\right) - \dfrac{v^2}{250}$ and find the glider's limiting speed (the speed it approaches but can't reach) in metres per second.

(c) Assuming the tensions in the ropes are constant and that the glider's initial speed is $50\,\text{m s}^{-1}$, find how far the glider travels before it reaches $70\,\text{m s}^{-1}$ and how long it takes.

16 A block of mass 5 kg is to be moved along a rough horizontal surface by a force of magnitude F newtons, inclined at an angle of θ to the direction of motion, $0 \le \theta \le \frac{\pi}{2}$.



There is a frictional force of magnitude R newtons, which is proportional to the normal reaction force of magnitude N, exerted on the block by the surface, such that $R = 0.2N$.

Take $g = 10 \text{ m s}^{-2}$.

(a) Show that when the block is about to move, $F = \dfrac{50}{5\cos\theta + \sin\theta}$ newtons.

(b) Calculate the minimum value of F needed to overcome the frictional resistance between the block and the surface.

17 A particle of unit mass moves in a horizontal straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, where $Q > 0$.

(a) Show that: $\tan^{-1}Q - \tan^{-1}v = \tan^{-1}\left[\dfrac{Q-v}{1+Qv}\right]$

(b) Show that $x = \tan^{-1}\left[\dfrac{Q-v}{1+Qv}\right]$, where x is the displacement.

(c) Show that $t = \dfrac{1}{2}\log_e\left[\dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}\right]$, where t is the elapsed time when the particle is travelling with velocity v.

(d) Find v^2 as a function of t.

(e) Find the limiting values of v and x as $t \to \infty$.

18 When a jet aircraft touches down, two different retarding forces combine to bring it to rest.

If the jet has a mass of M kg and a speed of $v \text{ m s}^{-1}$, there is a constant frictional force $\frac{1}{4}M$ newtons and a force of $\dfrac{1}{108}Mv^2$ newtons due to the reverse thrust of the engines.

The reverse thrust of the engines does not take effect until 20 seconds after the touchdown.

(a) Show that $\dfrac{d^2x}{dt^2} = -\dfrac{1}{4}$ for $0 \le t \le 20$, and that for $t > 20$, and until after it stops, $\dfrac{d^2x}{dt^2} = -\dfrac{1}{108}\left(27 + v^2\right)$.

(b) If the jet's speed at touchdown is 60 m s^{-1}, show that $v = 55$ and $x = 1150$ at the instant the reverse thrust of the engines takes effect.

(c) Show that when $t > 20$, $x = 1150 + 54\left\{\ln\left(27 + 55^2\right) - \ln\left(27 + v^2\right)\right\}$.

(d) Calculate how far from the touchdown point the jet comes to rest. Give your answer to the nearest metre.

19 A plane of mass 20 tonnes touches down on a runway while moving at a horizontal speed of 60 m s^{-1}. As the plane moves along the runway, its speed at time t seconds after touchdown is $v \text{ m s}^{-1}$. From the instant that it touches down, it is being slowed by a reverse thrust of 40 000 newtons supplied by the engines and a force of $4v^2$ newtons supplied by the braking effect of wing flaps and other frictional forces. After 10 seconds of motion along the runway, wheel brakes are applied which supply an additional $300(60 - v)$ newtons of braking force.

(a) Show that: $\ddot{x} = \begin{cases} -\left(2 + \dfrac{v^2}{5000}\right) & 0 \le t \le 10 \\ -\left(2.9 + \dfrac{v^2}{5000} - \dfrac{3v}{200}\right) & t > 10 \end{cases}$

(b) Find the speed of the plane 10 seconds after touchdown. Answer in m s^{-1} correct to one decimal place.

(c) Find the distance in metres (correct to one decimal place) travelled in the first 10 seconds after touchdown.

(d) Find the total time in seconds (correct to one decimal place) from touchdown until the plane stops. Without proof, you may assume $\int_0^{35.4} \frac{dv}{v^2 - 75v + 14500} = 0.00260721$ (correct to six significant figures).

(e) Find the total distance that the plane travels along the runway before it stops. Without proof, you may assume $\int_0^{35.4} \frac{v\,dv}{v^2 - 75v + 14500} = 0.0469319$ (correct to six significant figures).

(f) By integrating, verify the integral values given in parts **(d)** and **(e)**.

20 A car of mass m kg is being driven along a straight road. The engine of the car provides a constant propelling force mP, $P \in \mathbb{R}^+$, while the car experiences a resistive force of mkv^2, where $v\,\text{m s}^{-1}$ is the velocity of the car, and k is a positive constant. The car is initially at rest.

(a) By drawing a diagram, or otherwise, show that the situation described can be modelled by a differential equation of the form $\frac{dv}{dx} = g(v)$, where $g(v)$ is a function of v, and x is the displacement of the car from its initial position in metres.

(b) Show that $v^2 = \frac{P}{k}\left(1 - e^{-2kx}\right)$. Hence, or otherwise, explain why the maximum speed of the car is $v_M = \sqrt{\frac{P}{k}}\,\text{m s}^{-1}$.

(c) Show that the distance required for the car to reach a speed of $\frac{1}{3}v_M$ is approximately 41% of the distance required to reach a speed of $\frac{1}{2}v_M$.

6.8 VERTICAL RESISTED MOTION

Motion of a particle falling downwards in a resisting medium with gravity

Example 39

A body of mass 5 kg is dropped from a great height under a constant gravitational acceleration $g\,\text{m s}^{-2}$ and air resistance proportional to the speed $v\,\text{m s}^{-1}$. If the constant of proportionality is $\frac{1}{8}$:

(a) find the velocity at time t

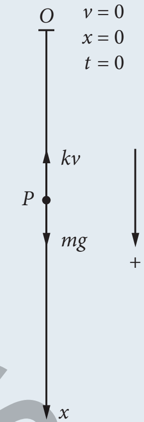
(b) sketch the velocity–time graph

(c) find the terminal (i.e. maximum) velocity

(d) find the distance the particle has fallen at time t.

Solution

(a) Take O as the point of release of the particle P. Take motion downwards as positive. There are two forces acting on P: its weight force of $5g$ acting downwards and its air resistance of $\frac{1}{8}v$ acting upwards (i.e. opposing the motion).



The resultant force on P is $5\ddot{x} = 5g - \frac{1}{8}v$ and this is the equation of motion:

$$\ddot{x} = g - \frac{v}{40}$$

$$\frac{dv}{dt} = \frac{40g - v}{40}$$

$$\frac{dt}{dv} = \frac{40}{40g - v}, \qquad v \neq 40g$$

$$\int_0^t dt = \int_0^v \frac{40}{40g - v}\,dv \qquad \text{(from } t = 0, v = 0 \text{ to } t = t, v = v\text{)}$$

$$t = -40\Big[\log_e(40g - v)\Big]_0^v, \qquad 0 \leq v < 40g$$

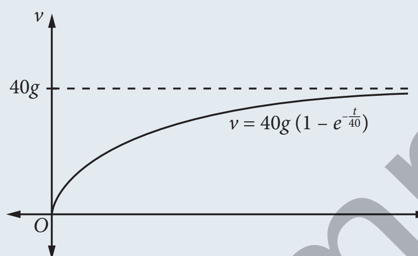
$$t = 40\log_e \frac{40g}{40g - v}$$

$$e^{\frac{t}{40}} = \frac{40g}{40g - v}$$

$$40g - v = 40ge^{-\frac{t}{40}}$$

$$v = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

(b)



(c) As t increases, the gravitational force acting on P causes it to accelerate downwards. However, as the speed increases, the resistance also increases until eventually the downwards weight force and the resistance force are equal in magnitude but opposite in direction. At this stage the resultant force is zero: according to Newton's first law of motion, P will now move in a straight line at constant speed, i.e. the terminal velocity.

The terminal velocity occurs when the acceleration is equal to zero.

Method 1 (graphically):
In the velocity–time graph in part **(b)**, the horizontal asymptote at $v = 40g$ indicates the terminal velocity.

Method 2 (algebraically):
As $\ddot{x} = g - \frac{v}{40}$ the terminal velocity occurs when $g - \frac{v}{40} = 0$, i.e. when $v = 40g$.

(d) $v = 40g\left(1 - e^{-\frac{t}{40}}\right)$

$$\frac{dx}{dt} = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

$$\int_0^x dx = 40g\int_0^t \left(1 - e^{-\frac{t}{40}}\right)dt \qquad \text{(from } x = 0, t = 0 \text{ to } x = x, t = t\text{)}$$

$$x = 40g\left[t + 40e^{-\frac{t}{40}}\right]_0^t$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}}\right) - 40g \times 40$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}} - 40\right)$$

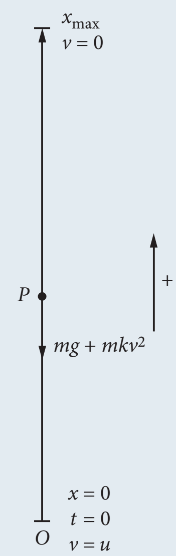
Motion of a particle moving upwards in a resisting medium with gravity

Example 40

A particle is projected vertically upwards with a velocity of u m s^{-1} under the influence of gravity and a resistance force proportional to the square of the speed. Find:

(a) the greatest height reached

(b) the time taken to reach the maximum height.



Solution

(a) Take O as the point of release of the particle P. Take motion upward as positive. There are two forces acting on P: its weight force of mg acting downwards and its resistance of mkv^2 acting downwards (i.e. opposing the motion).

(Note that although you do not have a numerical value for m, you write a factor of m into the constant of proportionality for the resistance. This will simplify later calculations where factors of m can be cancelled in the equation of motion.)

The resultant force on P is $m\ddot{x} = -mg - mkv^2$ and this is the equation of motion.
You need to find x when $v = 0$:

$$\ddot{x} = -\left(g + kv^2\right)$$

$$v\frac{dv}{dx} = -\left(g + kv^2\right)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\int_0^x dx = -\int_u^0 \frac{v}{g + kv^2}\,dv \quad \text{(from } x = 0, v = u \text{ to } x = x, v = 0\text{)}$$

$$x = -\frac{1}{2k}\left[\log_e\left(g + kv^2\right)\right]_u^0$$

$$x = -\frac{1}{2k}\log_e g + \frac{1}{2k}\log_e\left(g + ku^2\right)$$

$$x = \frac{1}{2k}\log_e\left(\frac{g + ku^2}{g}\right)$$

The maximum height attained has a value of: $\frac{1}{2k}\log_e\left(\frac{g + ku^2}{g}\right)$ m

(b) You now need to find t when $v = 0$:

$$\ddot{x} = -\left(g + kv^2\right)$$

$$\frac{dv}{dt} = -\left(g + kv^2\right)$$

$$\int_0^t dt = -\int_u^0 \frac{1}{g + kv^2}\,dv \quad \text{(from } t = 0, v = u \text{ to } t = t, v = 0\text{)}$$

$$t = -\frac{1}{k}\int_u^0 \frac{1}{\left(\frac{g}{k} + v^2\right)}\,dv$$

$$t = -\frac{1}{k} \times \sqrt{\frac{k}{g}}\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}v\right)\right]_u^0$$

$$t = -\frac{1}{\sqrt{kg}}\left(\tan^{-1}0 - \tan^{-1}\left(\sqrt{\frac{k}{g}}u\right)\right)$$

$t = \frac{1}{\sqrt{kg}}\tan^{-1}\left(\sqrt{\frac{k}{g}}u\right)$ seconds, this is the time taken to reach the maximum height.

The next example looks at the motion of a particle that goes upwards and then downwards. This requires the solution to take a new O and a new positive direction when you analyse the downwards motion.

Example 41

A particle of mass 1 kg is projected vertically upwards from the ground at a speed of V m s^{-1}. The particle is acted on by both gravity and a resistance of magnitude $0.02v^2$, where v is the velocity of the particle at time t.

- **(a)** Explain why the equation of motion while the particle is moving upwards is: $\ddot{x} = -\left(g + 0.02v^2\right)$
- **(b)** Find the greatest height h reached by the particle.
- **(c)** Find the time taken to reach this greatest height.

Having reached its maximum height, the particle falls back down towards its initial point of projection.

- **(d)** Write the equation of motion for the downwards journey.
- **(e)** Find the speed of the particle when it hits the ground.
- **(f)** Determine whether the particle's speed on return is less than, equal to, or greater than its initial speed on projection.
- **(g)** Find the time taken for the particle to hit the ground from its greatest height and hence state the time of flight for the particle.

Solution

(a) Take O as the point of projection and take upwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ downwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = -mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = -\left(g + 0.02v^2\right)$

(b)

$$\ddot{x} = -\left(g + 0.02v^2\right)$$

$$v\frac{dv}{dx} = -\left(\frac{50g + v^2}{50}\right)$$

$$\frac{dv}{dx} = -\left(\frac{50g + v^2}{50v}\right)$$

$$\frac{dx}{dv} = -\frac{50v}{50g + v^2}$$

$$\int_0^h dx = -\int_V^0 \frac{50v}{50g + v^2}\,dv \quad \text{(from } x = 0,\ v = V \text{ to } x = h,\ v = 0\text{)}$$

$$h = -25\left[\log_e\left(50g + v^2\right)\right]_V^0$$

$$h = -25\left(\log_e 50g - \log_e\left(50g + V^2\right)\right)$$

$$h = 25\log_e\left(\frac{50g + V^2}{50g}\right)$$

(c)

$$\frac{dv}{dt} = -\left(\frac{50g + v^2}{50}\right)$$

$$\frac{dt}{dv} = -\frac{50}{50g + v^2}$$

$$\int_0^t dt = -\int_V^0 \frac{50}{50g + v^2}\,dv \quad \text{(from } t = 0,\ v = V \text{ to } t = t,\ v = 0\text{)}$$

$$t = -\frac{50}{\sqrt{50g}}\left[\tan^{-1}\left(\frac{v}{\sqrt{50g}}\right)\right]_V^0$$

$$t = \sqrt{\frac{50}{g}}\tan^{-1}\left(\frac{V}{\sqrt{50g}}\right)$$

(d) For the downwards journey it is necessary to take a new O and a new positive direction. Remember:

> In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.
>
> This simplifies the calculations required to solve the problem.

Take O as the maximum height and take downwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ upwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = g - 0.02v^2$

(e) $v\dfrac{dv}{dx} = \dfrac{50g - v^2}{50}$

$$\frac{dv}{dx} = \frac{50g - v^2}{50v}$$

$$\frac{dx}{dv} = \frac{50v}{50g - v^2}$$

You need to find v when $x = h$:

$$\int_0^h dx = \int_0^v \frac{50v}{50g - v^2}\,dv$$

$$h = -25\left[\log_e\left|50g - v^2\right|\right]_0^v$$

$$h = 25\log_e\left|\frac{50g}{50g - v^2}\right|$$

But from **(b)**: $h = 25\log_e\left|\dfrac{50g + V^2}{50g}\right|$

$$\therefore\ 25\log_e\left|\frac{50g + V^2}{50g}\right| = 25\log_e\left|\frac{50g}{50g - v^2}\right|$$

$$\frac{50g + V^2}{50g} = \frac{50g}{50g - v^2}$$

$$2500g^2 + 50gV^2 - 50gv^2 - v^2V^2 = 2500g^2$$

$$50gV^2 = v^2(50g + V^2)$$

$v^2 = \dfrac{50gV^2}{50g + V^2}$ and so the speed on return is: $V\sqrt{\dfrac{50g}{50g + V^2}}$

(f) $V^2 > 0$, so $50g + V^2 > 50g$ and $\sqrt{\dfrac{50g}{50g + V^2}} < 1$

$\therefore$ Speed on return is less than V, i.e. the speed on return is less than the speed of projection.

(g) $\dfrac{dv}{dt} = \dfrac{50g - v^2}{50}$ from **(e)**

$$\frac{dt}{dv} = \frac{50}{50g - v^2}$$

$$t = \int_0^v \frac{50dv}{50 - v^2}$$

$$= \frac{50}{10\sqrt{2g}}\int_0^v\left(\frac{1}{5\sqrt{2g} + v} + \frac{1}{5\sqrt{2g} - v}\right)dv$$

$$= \frac{5}{\sqrt{2g}}\left[\ln\left|5\sqrt{2g} + v\right| - \ln\left|5\sqrt{2g} - v\right|\right]_0^v$$

$$= \frac{5}{\sqrt{2g}}\left(\ln\left|\frac{5\sqrt{2g} + v}{5\sqrt{2g} - v}\right| - 0\right)$$

$$= \frac{5}{\sqrt{2g}}\ln\left|\frac{5\sqrt{2g} + v}{5\sqrt{2g} - v}\right|$$

From (e) it hits the ground with a speed given by $v = V\sqrt{\dfrac{50g}{50g+V^2}}$.

Hence time of downward motion is given by $t = \dfrac{5}{\sqrt{2g}}\ln\left|\dfrac{5\sqrt{2g}+V\sqrt{\dfrac{50g}{50g+V^2}}}{5\sqrt{2g}-V\sqrt{\dfrac{50g}{50g+V^2}}}\right|$

Total time of flight is $\dfrac{10}{\sqrt{2g}}\tan^{-1}\left(\dfrac{V}{5\sqrt{2g}}\right)+\dfrac{5}{\sqrt{2g}}\ln\left|\dfrac{5\sqrt{2g}+V\sqrt{\dfrac{50g}{50g+V^2}}}{5\sqrt{2g}-V\sqrt{\dfrac{50g}{50g+V^2}}}\right|$ seconds.

NB: Not a nice answer, but the process is the important part.

EXERCISE 6.8 VERTICAL RESISTED MOTION

1 An object of mass m falls from rest under constant gravitational force and against air resistance equal to kv, where v is the speed and k is a positive constant.

(a) Find its velocity at any time t.
(b) Sketch the velocity–time graph.
(c) Find the terminal velocity. Find the time taken to reach a speed v_1 where v_1 is one-quarter of the terminal velocity.
(d) Find the distance travelled when the speed v_1 is reached.

2 A particle of mass 10 kg falls from rest and is subject to a force of $(98 - 2v)$ newtons, where v is the speed of the particle at time t seconds.

(a) In writing the force as '$(98 - 2v)$' newtons, the '98' represents the weight force of the particle. What is the physical meaning of the '$2v$'?
(b) Find the terminal velocity.
(c) Find the distance fallen in the first 5 seconds.

3 A particle falls from rest under constant gravity and a resistance force. If the retardation due to the resistance force varies as the square of the velocity, find:

(a) the equation of motion
(b) the terminal velocity
(c) the distance fallen as a function of the velocity
(d) the distance fallen when half the terminal velocity is reached
(e) the time taken to reach half the terminal velocity.

4 An object falls towards the Earth with constant gravitational acceleration g and against a resistance that produces retardation proportional to the velocity.

(a) State the equation of motion.
(b) Express the velocity v as a function of time t.
(c) State the terminal velocity.

If the constant of proportionality $k = 0.2$ and $g = 9.8$, find:

(d) the velocity after 5 seconds
(e) the time required to reach half the terminal velocity.

5 A particle is dropped from a height of 1000 metres in a medium whose resistance provides a retardation of $0.004v^2$, where v is the velocity.

(a) Find v^2 in terms of x, the distance fallen.
(b) Find the speed at which the particle reaches the lowest point.

6 A parachutist jumps from a stationary balloon at a great height. The parachute opens after 10 seconds. Assume that air resistance produces a retardation proportional to the velocity, with a constant of proportionality $k = 0.1$ for the first 10 seconds (i.e. during freefall) and $k = 2$ after the parachute opens. Find:

(a) the parachutist's velocity after 10 seconds **(b)** the parachutist's velocity after 15 seconds

(c) the parachutist's terminal velocity, i.e. the approximate velocity while floating to the ground.

7 A particle of unit mass is projected vertically upwards from the ground with initial velocity of u m s^{-1}.

The particle is affected by a gravitational force, g, and a resistive force equivalent to $\frac{1}{2}kv^2$, where the particles velocity is v m s^{-1}. Let k be the constant of proportionality and x be the displacement, in metres, from the ground after t seconds.

Which of the following is the maximum height, H, of the particle?

A $H = \int_u^0 \frac{v}{g + \frac{kv^2}{2}}\,dv$ **B** $H = \int_0^u \frac{v}{g + \frac{kv^2}{2}}\,dv$

C $H = \int_u^0 \frac{v}{g - \frac{kv^2}{2}}\,dv$ **D** $H = \int_0^u \frac{v}{g - \frac{kv^2}{2}}\,dv$

8 A particle is projected vertically upwards against air resistance. Its acceleration at any time t seconds after projection is given by $\ddot{x} = -\left(g + \frac{1}{10}v^2\right)$, where v m s^{-1} is the velocity. If the initial velocity is 20 m s^{-1}, find:

(a) the greatest height reached **(b)** the time taken to reach the greatest height.

9 A particle is projected vertically upwards with initial speed u. Its acceleration is given by the differential equation $\ddot{x} = -(g + kv)$ where v is the speed at any time t, k is a positive constant and kv is the retardation due to air resistance.

(a) Find the maximum height reached by the particle.

(b) Find the time taken to reach the maximum height.

(c) Write the differential equation for the downward motion.

(d) Show that the particle returns to its point of projection with a speed V given by: $k(u + V) = g\log_e\left[\frac{g + ku}{g - kV}\right]$

10 An object of mass m is projected vertically upwards with speed u. Air resistance is equal to k times the square of the speed, where k is a positive constant.

(a) Find the maximum height reached by the object.

(b) Find the speed V of the object when it returns to the point of projection.

(c) Show that $V < u$.

11 An object, projected vertically upwards with speed U, returns to the point of projection with speed V. Assuming constant gravity and air resistance proportional to the square of the speed, find the total time taken in terms of U and V.

12 A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{10}mv^2$ newtons, where the velocity of the particle is v m s^{-1}. After t seconds, the particle has fallen x metres. Take $g = 10$ m s^{-2}.

(a) With the aid of a diagram, show that the acceleration is $\ddot{x} = \frac{1}{10}\left(100 - v^2\right)$.

(b) The particle hits the ground $\frac{1}{2}\ln 5$ seconds after it is dropped. Find the distance fallen in simplest exact form.

13 A body of unit mass falls vertically from rest, under gravity, from a height of 100 metres above the ground through a resisting medium. The resistance to its motion is $\frac{v^2}{100}$, where v is the velocity of the body, in m s^{-1}, after falling a distance of x metres.

(a) Show that the acceleration, $\ddot{x}$, of the body is given by $\ddot{x} = g - \frac{v^2}{100}$.

(b) Determine the terminal velocity, V, of the body.

(c) Show that the velocity, v, of the body, after it has fallen a vertical distance, x, is given by $v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$.

(d) Determine the distance fallen when the velocity of the body first reaches $\frac{V}{2}$.

14 A unit mass is projected vertically upwards under gravity with speed S metres per second in a medium that exerts a resistance to the motion equal to kv newtons, where k is a positive constant and v is the upward velocity of the mass in metres per second.

The acceleration due to gravity is $g\ \text{m s}^{-2}$, where $g > 0$.

If the particle reaches its greatest height in time T seconds, show for $0 \le t \le T$, that $v = \frac{g\left(e^{k(T-t)} - 1\right)}{k}$.

15 A particle of unit mass is projected vertically upwards in a medium in which the retardation due to resistance is $0.1v$. It is allowed to fall back to its point of projection. The initial speed of projection is V_0 and the final speed on return is V_F. Show that:

(a) the equation of motion on the upwards journey is $\ddot{x} = -(g + 0.1v)$

(b) the maximum height reached is $h = 10V_0 + 100g\log_e\left(\frac{10g}{10g + V_0}\right)$

(c) the time taken to reach the highest point is $T_1 = 10\log_e\left(\frac{10g + V_0}{10g}\right)$

(d) the equation of motion on the downwards journey is $\ddot{x} = g - 0.1v$

(e) the time taken on the downwards journey is $T_2 = 10\log_e\left(\frac{10g}{10g - V_F}\right)$

(f) by analysis of the downwards journey, $h = -10V_F + 100g\log_e\left(\frac{10g}{10g - V_F}\right)$

(g) the total time of the motion is $T = \frac{V_0 + V_F}{g}$.

16 Allyn decides to go bungy-jumping. This involves being tied to a bridge at point O by an elastic cable of length L metres, then falling vertically from rest from this point.

After Allyn free-falls L metres he is slowed down by the cable, which exerts a force (in newtons) of mkg times the distance greater than L that he has fallen, where m is his mass in kg and k is a constant.

Let x metres be the distance Allyn has fallen, and let $v\ \text{m s}^{-1}$ be his speed at x.

(a) Show that $\ddot{x} = \begin{cases} g & x \le L \\ g - gk(x - L) & x > L \end{cases}$

(b) Show that $v^2 = 2gL$ at the instant Allyn is at $x = L$.

(c) Show that $v^2 = 2gx - kg(x - L)^2$ for $x > L$.

(d) Show that Allyn stops for the first time at $x = L + \frac{1}{k} + \sqrt{\frac{2L}{k} + \left(\frac{1}{k}\right)^2}$.

(e) Given that $\frac{1}{k} = \frac{L}{4}$, show that O must be at least $2L$ metres above any obstruction on Allyn's path.

6.9 PROJECTILES AND RESISTED MOTION

Review of projectile motion with no resistance

In *New Senior Mathematics Extension 1* you looked at projectile motion. Unless otherwise stated, any air resistance is ignored when investigating projectile motion and it is assumed that the only force acting on the projectile is the force of gravity. Furthermore, it is assumed that the gravitational force is constant throughout the motion. The following examples **42**–**44** review projectile motion with no resistance.

Use $\underset{\sim}{\ddot{r}}$ for the acceleration vector, $\underset{\sim}{\dot{r}}$ for the velocity vector and $\underset{\sim}{r}$ for the displacement vector.

Example 42

A golf ball of mass m kg is hit down the middle of a fairway with an initial speed of $25\,\text{m s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \frac{4}{3}$.

(a) Taking unit vectors $\underset{\sim}{i}$ horizontally in the direction of motion and $\underset{\sim}{j}$ vertically upward, find an expression for the initial velocity $\underset{\sim}{u}\ \text{m s}^{-1}$ of the ball.

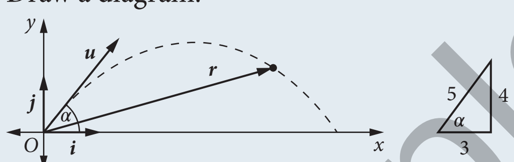
(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the ball after t seconds.

(c) Assuming the fairway to be horizontal, find the horizontal distance that the ball travels before hitting the ground. Give your answer correct to one decimal place.

(d) Find the maximum height reached by the ball during its flight. Give your answer correct to one decimal place.

Solution

(a) Draw a diagram.



Express $\underset{\sim}{u}$ in component form: $\underset{\sim}{u} = u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j}$

$$= 25\left(\frac{3}{5}\right)\underset{\sim}{i} + 25\left(\frac{4}{5}\right)\underset{\sim}{j}$$

$$= 15\underset{\sim}{i} + 20\underset{\sim}{j}$$

(b) Determine the equation of motion: If the ball has a mass of m kg, its equation of motion is $-mg\,\underset{\sim}{j} = m\underset{\sim}{\ddot{r}}\quad \therefore \underset{\sim}{\ddot{r}} = -g\,\underset{\sim}{j}$

Integrate each component with respect to t to find $\underset{\sim}{\dot{r}}$: $\quad \underset{\sim}{\dot{r}} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$

Apply the initial condition to find $\underset{\sim}{c}$: $\quad$ At $t = 0$, $\underset{\sim}{\dot{r}} = \underset{\sim}{u}$ so $\underset{\sim}{c} = 15\underset{\sim}{i} + 20\underset{\sim}{j}$

Hence $\underset{\sim}{\dot{r}} = 15\underset{\sim}{i} + (20 - gt)\underset{\sim}{j}$

Integrate each component with respect to t to find $\underset{\sim}{r}$: $\quad \underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

Apply the initial condition to find $\underset{\sim}{d}$: $\quad$ At $t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$ so $\underset{\sim}{d} = \underset{\sim}{0}$

Hence $\underset{\sim}{r} = 15t\underset{\sim}{i} + \left(20t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$

(c) The ball hits the ground when the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$ is zero: $20t - 4.9t^2 = 0$

$$t(20 - 4.9t) = 0$$

$$t = 0 \text{ or } t = \frac{20}{4.9} \approx 4.08\,\text{s}$$

The horizontal distance that the ball has covered is the $\underset{\sim}{i}$ component of $\underset{\sim}{r}$:

$$\begin{aligned}\text{Distance} &= 15t \\ &= 15 \times \frac{20}{4.9} \\ &= 61.2\,\text{m}\end{aligned}$$

The ball travels 61.2 m before hitting the ground.

(d) The ball reaches its maximum height when the $\underset{\sim}{j}$ component of $\dot{\underset{\sim}{r}}$ is zero: $20 - 9.8t = 0$

$$t = \frac{20}{9.8}$$

$$t = 2.04\,\text{s}$$

Its height is given by the $\underset{\sim}{j}$ component of $\underset{\sim}{r}$:

$$\begin{aligned}h_{max} &= h(2.04) \\ &= 2.04\left(20 - 4.9 \times \frac{20}{9.8}\right) \\ &= 2.04 \times 10 \\ &= 20.4\,\text{m}\end{aligned}$$

The ball reaches a maximum height of 20.4 m.

Example 43

A football is kicked towards goal with an initial speed of $26\,\text{m s}^{-1}$ at an angle of projection to the horizontal α, where $\tan\alpha = \frac{5}{12}$.

(a) Find an expression for the initial velocity $\underset{\sim}{u}\,\text{m s}^{-1}$ of the football.

(b) Taking the origin at the point of projection, find an expression for the position vector $\underset{\sim}{r}$ of the football after t seconds.

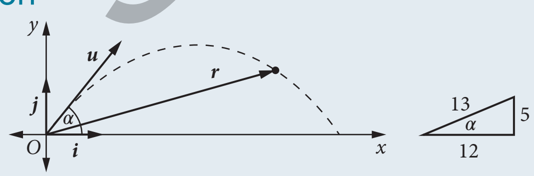
(c) Write the parametric equations of the path of the football and use them to find the Cartesian equation of the path.

(d) Assuming the ground to be horizontal, find the horizontal distance that the football travels before hitting the ground. Give your answer correct to the nearest metre.

(e) The horizontal crossbar of the goal is 2.44 m above the ground and is 0.1 m deep. If the football is heading towards the goal, which is 40 m from where the football is kicked, will the football pass below the crossbar?

Solution

(a)



$u = 26\,\text{m s}^{-1}$, $\tan\alpha = \frac{5}{12}$ so $\sin\alpha = \frac{5}{\sqrt{5^2 + 12^2}} = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$

$$\begin{aligned}\underset{\sim}{u} &= u\cos\alpha\,\underset{\sim}{i} + u\sin\alpha\,\underset{\sim}{j} \\ &= 26 \times \frac{12}{13}\underset{\sim}{i} + 26 \times \frac{5}{13}\underset{\sim}{j} \\ &= 24\underset{\sim}{i} + 10\underset{\sim}{j}\end{aligned}$$

(b) $\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j}$ as gravity is acting against the motion in the vertical direction.

Integrate with respect to t: $\quad \dot{\underset{\sim}{r}} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0$, $\underset{\sim}{u} = 24\underset{\sim}{i} + 10\underset{\sim}{j}$: $\quad 24\underset{\sim}{i} + 10\underset{\sim}{j} = \underset{\sim}{c}$

$$\begin{aligned}\dot{\underset{\sim}{r}} &= -gt\,\underset{\sim}{j} + 24\underset{\sim}{i} + 10\underset{\sim}{j}\\ &= 24\underset{\sim}{i} + (10 - gt)\underset{\sim}{j}\end{aligned}$$

Integrate with respect to t: $\quad \underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \dfrac{gt^2}{2}\right)\underset{\sim}{j} + \underset{\sim}{d}$

$t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$: $\qquad \underset{\sim}{d} = \underset{\sim}{0}$

$$\underset{\sim}{r} = 24t\underset{\sim}{i} + \left(10t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$$

(c) $x = 24t$, $y = 10t - \dfrac{1}{2}gt^2$ are the parametric equations of the path.

$$t = \frac{x}{24},\ y = \frac{10x}{24} - \frac{g}{2} \times \left(\frac{x}{24}\right)^2$$

If $g = 9.8\,\text{m s}^{-2}$: $\quad y = \dfrac{10x}{24} - \dfrac{4.9x^2}{576}$ is the Cartesian equation of the path.

(d) Hits the ground when $y = 0$:

$$\begin{aligned}\frac{10x}{24} - \frac{4.9x^2}{576} &= 0\\ x(240 - 4.9x) &= 0\\ x = 0 \text{ or } x = \frac{240}{4.9} &\approx 49\,\text{m}\end{aligned}$$

The ball hits the ground 49 metres from where it was kicked.

(e) $x = 40$, $y = \dfrac{10}{24} \times 40 - \dfrac{4.9}{576} \times 40^2$

$\approx 3.06\,\text{m}$

The ball will pass over the crossbar.

Example 44

A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}\,\text{m s}^{-1}$, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up. Use $g = 9.8\,\text{m s}^{-2}$.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection correct to the nearest tenth of a degree.

(b) Find the time of flight of the particle. Give your answer correct to one decimal place.

(c) Find the horizontal distance travelled by the particle, correct to one decimal place.

(d) Find the maximum height reached by the particle, correct to one decimal place.

(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

Solution

(a) $\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad |\underset{\sim}{u}| = \sqrt{7^2 + 24^2} = 25\,\text{m s}^{-1}$

Angle of projection θ is $\theta = \tan^{-1}\left(\frac{24}{7}\right) = 73.7^\circ$ to the horizontal

(b) $\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j}$

Integrate with respect to t: $\quad \dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + \underset{\sim}{c}$

$t = 0, \underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$: $\quad 7\underset{\sim}{i} + 24\underset{\sim}{j} = \underset{\sim}{c}$

$\dot{r} = -gt\underset{\sim}{j} + 7\underset{\sim}{i} + 24\underset{\sim}{j}$

$= 7\underset{\sim}{i} + (24 - gt)\underset{\sim}{j}$

Integrate with respect to t: $\quad \underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \frac{1}{2}gt^2\right)\underset{\sim}{j} + \underset{\sim}{d}$

$\underset{\sim}{r}(0) = \underset{\sim}{0}$: $\quad \underset{\sim}{d} = \underset{\sim}{0}$

$\underset{\sim}{r} = 7t\underset{\sim}{i} + \left(24t - \frac{1}{2}gt^2\right)\underset{\sim}{j} = 7t\underset{\sim}{i} + \left(24t - 4.9t^2\right)\underset{\sim}{j}$

For the time of flight, find when the vertical component of $\underset{\sim}{r}$ is zero: $24t - 4.9t^2 = 0$

$$t(24 - 4.9t) = 0$$

$$t = 0, t = \frac{24}{4.9} = 4.9\,\text{s}$$

Hence the time of flight is 4.9 seconds.

(c) Substitute $t = 4.9$ into the horizontal component of $\underset{\sim}{r}$: Distance $= 7 \times 4.9 = 34.3\,\text{m}$

(d) The time to the greatest height is half the time of flight: $t = 2.45$

Substitute this value of t into the vertical component of $\underset{\sim}{r}$:
Greatest height $= 24 \times 2.45 - 4.9 \times 2.45^2 \approx 29.4\,\text{m}$

(e) The direction at any instant is given by the velocity function as it is tangential to the position function at any point in the path.

$\underset{\sim}{u} = 7\underset{\sim}{i} + 24\underset{\sim}{j}$, $\dot{\underset{\sim}{r}} = 7\underset{\sim}{i} + (24 - 9.8t)\underset{\sim}{j}$. Find when $\underset{\sim}{u} \bullet \dot{\underset{\sim}{r}} = 0$: $\left(7\underset{\sim}{i} + 24\underset{\sim}{j}\right) \bullet \left(7\underset{\sim}{i} + \left(24 - 9.8t\right)\underset{\sim}{j}\right) = 0$

$$49 + 24(24 - 9.8t) = 0$$

$$49 + 576 - 235.2t = 0$$

$$t = \frac{625}{235.2} = 2.66\,\text{s}$$

The particle is travelling in a direction perpendicular to the original direction at 2.66 seconds.

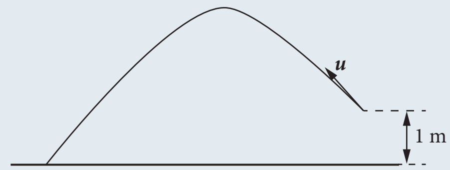
As considered in section 6.5, projectile motion with no air resistance is generally not a realistic model because real projectiles experience resistance due to the medium that they move through. This resistance is a force that can often be approximated as being proportional to the velocity, or proportional to the square of the velocity of the projectile, depending on the conditions of the medium. In this section you will consider resistance proportional to the velocity of the projectile.

Resistance as well as gravity—a vector approach

If there is any resistance other than gravity, it needs to be incorporated into the acceleration vector. The new acceleration vector can then be used to form the other equations of motion.

Example 45

During a game of badminton at the beach, a shuttlecock is hit at a height of 1 m with a velocity of $2\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$ m s^{-1}, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively. The acceleration of the shuttlecock due to the combined effect of gravity, air resistance and wind is $2\underset{\sim}{i}-\underset{\sim}{j}-8\underset{\sim}{k}$ m s^{-2}. Assume the sand to be horizontally flat and take the origin to be at sand level, directly below the point of projection.



(a) Find the time of flight of the shuttlecock. Give your answer correct to two decimal places.

(b) Find where the shuttlecock will land.

(c) Find the maximum height reached by the shuttlecock.

Solution

(a) Define $\ddot{\underset{\sim}{r}}$: $\ddot{\underset{\sim}{r}} = 2\underset{\sim}{i}-\underset{\sim}{j}-8\underset{\sim}{k}$.

Integrate with respect to t: $\dot{\underset{\sim}{r}} = \int \ddot{\underset{\sim}{r}}\,dt$

$$= \int \left(2\underset{\sim}{i}-\underset{\sim}{j}-8\underset{\sim}{k}\right)dt$$

$$= 2t\underset{\sim}{i}-t\underset{\sim}{j}-8t\underset{\sim}{k}+\underset{\sim}{c}$$

$t=0$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$: $2\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}=\underset{\sim}{c}$

Hence $\dot{\underset{\sim}{r}} = 2t\underset{\sim}{i}-t\underset{\sim}{j}-8t\underset{\sim}{k}+2\underset{\sim}{i}+2\underset{\sim}{j}+8\underset{\sim}{k}$

$$= (2+2t)\underset{\sim}{i}+(2-t)\underset{\sim}{j}+(8-8t)\underset{\sim}{k}$$

Integrate with respect to t: $\underset{\sim}{r} = \int \left((2+2t)\underset{\sim}{i}+(2-t)\underset{\sim}{j}+(8-8t)\underset{\sim}{k}\right)dt$

$$= \left(2t+t^2\right)\underset{\sim}{i}+\left(2t-\frac{t^2}{2}\right)\underset{\sim}{j}+\left(8t-4t^2\right)\underset{\sim}{k}+\underset{\sim}{d}$$

$t=0$, $\underset{\sim}{r}=\underset{\sim}{k}$: $\underset{\sim}{k}=\underset{\sim}{d}$

Hence $\underset{\sim}{r} = \left(2t+t^2\right)\underset{\sim}{i}+\left(2t-\frac{t^2}{2}\right)\underset{\sim}{j}+\left(1+8t-4t^2\right)\underset{\sim}{k}$

The shuttlecock hits the sand when the vertical component ($\underset{\sim}{k}$) of the motion is zero: $1+8t-4t^2=0$

Rewrite as $4t^2-8t-1=0$ and solve: $t=\frac{8\pm\sqrt{80}}{8}=\frac{2\pm\sqrt{5}}{2}$

As $t>0$, the only solution is $t=\frac{2+\sqrt{5}}{2}\approx 2.12$ s

The time of flight of the shuttlecock is 2.12 seconds.

(b) Find $\underset{\sim}{r}$ when $t=2.12$ s: $\underset{\sim}{r} = \left(4.24+2.12^2\right)\underset{\sim}{i}+\left(4.24-\frac{2.12^2}{2}\right)\underset{\sim}{j}+0\underset{\sim}{k}$

$$= 8.73\underset{\sim}{i}+1.99\underset{\sim}{j}$$

The shuttlecock lands approximately 8.7 m east and 2 m north of its point of projection.

(c) The maximum height is achieved when the $\underset{\sim}{k}$ component of the velocity vector is zero: $8-8t=0$

$t=1$ s

Hence the maximum height can be found by substituting $t=1$ into the $\underset{\sim}{k}$ component of the displacement: $h=1+8(1)-4(1)^2=5$ m

The shuttlecock reaches a maximum height of 5 metres.

When the resistance to the motion is included in the acceleration vector, the solution to the problem follows the same approach as used earlier. Note that if you are working in three dimensions, coordinates will be needed to locate points in a plane.

Resistance as well as gravity—a Cartesian approach

A particle of mass m is launched at time $t = 0$, from ground level on a flat plane, with an initial velocity of u m s^{-1} at an angle of θ to the horizontal. In addition to gravity, there is an air resistance force, which acts in the opposite direction to the instantaneous direction of motion. The magnitude of this resistance force is directly proportional to the particle's instantaneous speed.

Use standard Cartesian coordinates with the x-axis horizontal and the y-axis vertical. Let the components of the acceleration be $\ddot{x}$ and $\ddot{y}$, so that the components of the velocity are $\dot{x}$ and $\dot{y}$ and so the components of the displacement are x and y.

Initially: $x = 0$, $\dot{x} = u_x = u\cos\theta$, $y = 0$, $\dot{y} = u_y = u\sin\theta$.

Now $m\ddot{x} = -mk\dot{x}$ and $m\ddot{y} = -mg - mk\dot{y}$, where k is a positive constant.

Dividing by m reduces these equations to: $\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

Consider the horizontal motion and let $v_x = \dot{x}$ so that $\ddot{x} = \dfrac{dv_x}{dt}$:

$$\frac{dv_x}{dt} = -kv_x$$

$$\frac{dv_x}{v_x} = -kdt$$

Integrate with respect to t:

$$\int_{u_x}^{v_x} \frac{dv_x}{v_x} = -k\int_0^t dt$$

$$\left[\log_e(v_x)\right]_{u_x}^{v_x} = -kt$$

$$\log_e\left(\frac{v_x}{u_x}\right) = -kt$$

$$v_x = u_x e^{-kt}$$

Now $v_x = \dot{x}$: $\dfrac{dx}{dt} = u_x e^{-kt}$

Integrate with respect to t: $x = u_x \displaystyle\int_0^t e^{-kt} dt$

$$x = -\frac{u_x}{k}\left[e^{-kt}\right]_0^t$$

$$x = -\frac{u_x}{k}\left(e^{-kt} - 1\right)$$

$$x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$$

Consider the vertical motion and let $v_y = \dot{y}$ so that $\ddot{y} = \dfrac{dv_y}{dt}$:

$$\frac{dv_y}{dt} = -g - kv_y$$

$$\frac{dv_y}{g + kv_y} = -dt$$

Integrate with respect to t:
$$\int_{u_y}^{v_y} \frac{dv_y}{g+kv_y} = -\int_0^t dt$$
$$\left[\frac{1}{k}\log_e\left(g+kv_y\right)\right]_{u_y}^{v_y} = -t$$
$$\log_e\left(g+kv_y\right) - \log_e\left(g+ku_y\right) = -kt$$
$$\log_e\left(\frac{g+kv_y}{g+ku_y}\right) = -kt$$
$$\frac{g+kv_y}{g+ku_y} = e^{-kt}$$
$$kv_y = \left(g+ku_y\right)e^{-kt} - g$$
$$v_y = \frac{1}{k}\left(\left(g+ku_y\right)e^{-kt} - g\right)$$

Now $v_y = \dot{y}$ and $\frac{dy}{dt} = \frac{1}{k}\left(\left(g+ku_y\right)e^{-kt} - g\right)$.

Integrate with respect to t: $y = \frac{1}{k}\int_0^t \left(\left(g+ku_y\right)e^{-kt} - g\right)dt$
$$= \frac{1}{k}\left[\frac{g+ku_y}{-k}e^{-kt} - gt\right]_0^t$$
$$= -\frac{g+ku_y}{k^2}e^{-kt} - \frac{gt}{k} + \frac{g+ku_y}{k^2}$$
$$= \frac{g+ku_y}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k}$$

As a result of all of this, you have obtained the parametric equations of the velocity: $\dot{x} = u_x e^{-kt}$, $\dot{y} = \frac{1}{k}\left(\left(g+ku_y\right)e^{-kt} - g\right)$.

The parametric equations of the path are $x = \frac{u_x}{k}\left(1-e^{-kt}\right)$, $y = \frac{\left(g+ku_y\right)}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k}$.

Summary of important results

Projectile motion in a medium whose resistance is proportional to the velocity of the particle:

$t=0$, $x=0$, $y=0$, $\dot{x}=u_x$, $\dot{y}=u_y$ where $u_x = u\cos\theta$, $u_y = u\sin\theta$, k is the constant of proportionality in the resistance, θ is the angle of projection relative to the horizontal axis, and u is the initial velocity.

$\ddot{x} = -k\dot{x}$, $\ddot{y} = -g - k\dot{y}$

$\dot{x} = u_x e^{-kt} = u\cos\theta\, e^{-kt}$, $\dot{y} = \frac{1}{k}\left(\left(g+ku_y\right)e^{-kt} - g\right) = \frac{1}{k}\left((g+ku\sin\theta)e^{-kt} - g\right) = u\sin\theta\, e^{-kt} - \frac{g}{k}\left(1-e^{-kt}\right)$

$x = \frac{u_x}{k}\left(1-e^{-kt}\right) = \frac{u\cos\theta}{k}\left(1-e^{-kt}\right)$, $y = \frac{\left(g+ku_y\right)}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k} = \frac{\left(g+ku\sin\theta\right)}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k}$

With no air resistance, the equations are instead:

Acceleration: $\ddot{x} = 0$, $\ddot{y} = -g$

Velocity: $\dot{x} = u\cos\theta$, $\dot{y} = u\sin\theta - gt$

Displacement: $x = u\cos\theta t$, $y = u\sin\theta t - \frac{1}{2}gt^2$

When there is no air resistance, the horizontal velocity is $\dot{x} = u\cos\theta$, whereas with air resistance the horizontal velocity becomes $\dot{x} = u\cos\theta\, e^{-kt}$, which means the horizontal velocity is reducing over time, decaying exponentially. This means that with air resistance, the particle slows down and will not travel as far.

When there is no air resistance, the vertical velocity is $\dot{y} = u\sin\theta - gt$, whereas with air resistance the vertical velocity becomes $\dot{y} = u\sin\theta\, e^{-kt} - \frac{g}{k}\left(1 - e^{-kt}\right)$. With no air resistance the greatest height is achieved after

$\frac{u\sin\theta}{g}$ seconds, but with air resistance it is after $\frac{1}{k}\log_e\left|\frac{ku\sin\theta}{g} + 1\right|$ seconds.

With no air resistance, the greatest height attained is $\frac{u^2\sin^2\theta}{2g}$ m; with air resistance it is $\frac{u\sin\theta}{k} - \frac{g}{k^2}\log_e\left|1 + \frac{ku\sin\theta}{g}\right|$, which is always less (for sensible positive values of the constants).

This air resistance behaviour is best considered using a numerical example.

Example 46

A particle is projected from a point on the horizontal plane with an initial velocity given by the components $u_x = 3\,\text{m s}^{-1}$, $u_y = 8\,\text{m s}^{-1}$. Use $g = 9.8\,\text{m s}^{-2}$.

(a) If the only resistance to the motion is gravity, find the parametric equations of the trajectory of the particle. When does the particle hit the ground?

The particle is projected again into a medium which resists the motion, where the resistance to the motion is directly proportional to the velocity of the particle. Let the constant of proportionality be k.

The parametric equations of the trajectory are given as $x = \frac{u_x}{k}\left(1 - e^{-kt}\right)$, $y = \frac{\left(g + ku_y\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$.

(b) Find the parametric equations of the trajectory when **(i)** $k = 0.5$ **(ii)** $k = 0.1$.

(c) Using technology, draw on the same set of axes the graphs of the three trajectories in parts **(a)** and **(b)**.

(d) Discuss what your graphs in part **(c)** tell you about the motion in each case.

Solution

(a) No air resistance: $u_x = 3$, $u_y = 8$, $g = 9.8$.

Initial equations of motion:	$\ddot{x} = 0$	$\ddot{y} = -g = -9.8$
Integrate with respect to t:	$\dot{x} = C_1$	$\dot{y} = -9.8t + C_2$
$t = 0$, $u_x = 3$, $u_y = 8$:	$\dot{x} = 3$	$\dot{y} = 8 - 9.8t$
Integrate with respect to t:	$x = 3t + C_3$	$y = 8t - 4.9t^2 + C_4$
$t = 0$, $x = 0$, $y = 0$:	$x = 3t$	$y = 8t - 4.9t^2$

Hits the ground when $y = 0$: $8t - 4.9t^2 = 0$

$$t = \frac{8}{4.9}\text{ s}$$

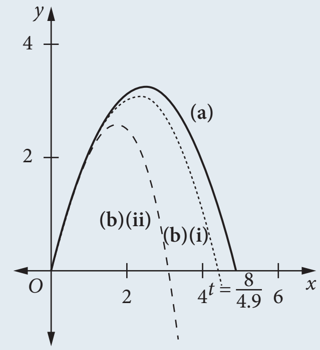
(b) **(i)** $k = \frac{1}{2}$: $x = \frac{3}{0.5}\left(1 - e^{-\frac{t}{2}}\right)$ $\qquad y = \frac{9.8 + 4}{0.5^2}\left(1 - e^{-\frac{t}{2}}\right) - \frac{9.8t}{0.5}$

$x = 6\left(1 - e^{-\frac{t}{2}}\right)$ $\qquad y = 55.2\left(1 - e^{-\frac{t}{2}}\right) - 19.6t$

(ii) $k = 0.1$: $x = \frac{3}{0.1}\left(1 - e^{-\frac{t}{10}}\right)$ $\qquad y = \frac{9.8 + 0.8}{0.1^2}\left(1 - e^{-\frac{t}{10}}\right) - \frac{9.8t}{0.1}$

$x = 30\left(1 - e^{-\frac{t}{10}}\right)$ $\qquad y = 1060\left(1 - e^{-\frac{t}{10}}\right) - 98t$

(c) In the diagram, the solid line represents part **(a)**, the dotted line represents part **(b)(i)** and the dashed line represents part **(b)(ii)**. Each line has been completed for $t = \frac{8}{4.9}$ seconds, the time taken for the particle without air resistance to return to the ground.



Two of the lines show $y < 0$, but this is just so the line lengths can be compared for the largest value of t in part **(a)**. After each particle hits the ground, it stops, so the parts of the paths below zero do not actually exist.

(d) At the beginning of the motion the paths are similar. The trajectory in part **(a)** is symmetrical about its greatest height.

The larger the air resistance, the sooner the trajectory falls below the path in part **(a)**. The particles in part **(b)** hit the ground before the particle in part **(a)**.

The larger the air resistance, the shorter the range.

The larger the air resistance, the lower the greatest height and the steeper the fall after the particle reaches its greatest height.

EXPLORING FURTHER

Projectile motion and initial velocity

Use technology to observe the effect of changing the initial velocity on a model of projectile motion.

Example 47

A projectile is fired from the origin O with an initial velocity u m s^{-1} at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \frac{u\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \frac{(10+ku\sin\theta)}{k^2}\left(1-e^{-kt}\right) - \frac{10t}{k}$,

where k is the constant of proportionality of the resistance.

The projectile is fired at an angle of 60° to the horizontal with an initial velocity of $10\sqrt{3}$ m s^{-1}, $k = 0.4$.

(a) Find when the projectile reaches its greatest height, correct to two decimal places.

(b) Find the greatest height that is reached, correct to two decimal places.

(c) Show that the projectile hits the ground when $t \approx 2.6$ s **(i)** graphically **(ii)** by substitution.

(d) Find the horizontal range of the projectile.

(e) Graph the path of the projectile.

Solution

Write the parametric equations of the trajectory using the information provided.

$$x = \frac{10\sqrt{3}\times\frac{1}{2}}{0.4}\left(1-e^{-0.4t}\right) = 12.5\sqrt{3}\left(1-e^{-0.4t}\right)$$

$$y = \frac{10+0.4\times10\sqrt{3}\times\frac{\sqrt{3}}{2}}{0.4^2}\left(1-e^{-0.4t}\right) - \frac{10t}{0.4} = 100\left(1-e^{-0.4t}\right) - 25t$$

(a) Greatest height when $\dot{y} = 0$: $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$\dot{y} = 40e^{-0.4t} - 25$$

$$40e^{-0.4t} - 25 = 0$$

$$e^{-0.4t} = \frac{5}{8}$$

$$-0.4t = \log_e\left(\frac{5}{8}\right)$$

$$t = 2.5\log_e 1.6$$

$$t = 1.18\,\text{s}$$

(b) $t = 1.18$: $y = 100\left(1 - e^{-0.4\times1.18}\right) - 25\times1.18$

$$= 8.12\,\text{m}$$

The greatest height is 8.12 metres.

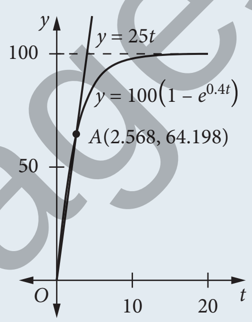
(c) **(i)** Hits the ground when $y = 0$: $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$100\left(1 - e^{-0.4t}\right) - 25t = 0$$

$$100\left(1 - e^{-0.4t}\right) = 25t$$

Hence graph $y = 100\left(1 - e^{-0.4t}\right)$ and $y = 25t$.

It hits the ground at $t = 2.6$ seconds.



(ii) $t = 2.6$ s, $y = 100\left(1 - e^{-0.4t}\right) - 25t$

$$= 100\left(1 - e^{-1.04}\right) - 25\times2.6$$

$$= -0.345 \approx 0$$

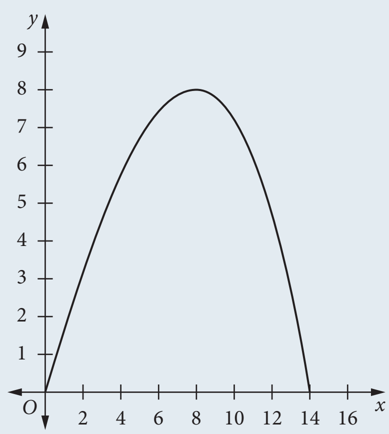
It hits the ground at approximately $t = 2.6$ seconds.

(d) $t = 2.6$, $x = 12.5\sqrt{3}\left(1 - e^{-0.4\times2.6}\right)$

$$= 12.5\sqrt{3}\left(1 - e^{-1.04}\right)$$

$$= 13.998 \approx 14.0\,\text{m}$$

(e)



Using terminal velocity

Given the terminal velocity $v_T = \frac{g}{k}$, consider what this means in the equations involving air resistance. Remember, the terminal velocity for a projectile falling vertically downwards is the velocity at which the drag force (air resistance) balances the gravitational force.

Now $v_T = \frac{g}{k}$, $k = \frac{g}{v_T}$ or $\frac{1}{k} = \frac{v_T}{g}$ can be substituted in each of the previous equations for air resistance motion.

Horizontally: $\ddot{x} = -kx = -\frac{g\dot{x}}{v_T}$, $\dot{x} = u_x e^{-kt} = u_x e^{\frac{-gt}{v_T}}$, $x = \frac{u_x}{k}\left(1-e^{-kt}\right) = \frac{u_x v_T}{g}\left(1-e^{\frac{-gt}{v_T}}\right)$.

When there is no air resistance, the horizontal velocity is $\dot{x} = u_x$, whereas with air resistance the horizontal velocity has become $\dot{x} = u_x e^{\frac{-gt}{v_T}}$, which is reducing over time, decaying exponentially.

Vertically: $\ddot{y} = -g - k\dot{y} = -g\left(1 + \frac{\dot{y}}{v_T}\right)$,

$$\dot{y} = \frac{1}{k}\left(\left(g + ku_y\right)e^{-kt} - g\right) = \left(v_T + u_y\right)e^{\frac{-gt}{v_T}} - v_T = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right),$$

$$y = \frac{g + ku_y}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k} = \frac{v_T}{g}\left(v_T + u_y\right)\left(1 - e^{\frac{-gt}{v_T}}\right) - v_T t.$$

When there is no air resistance, the vertical velocity is $\dot{y} = u_y - gt$, whereas with air resistance the vertical velocity has become $\dot{y} = u_y e^{\frac{-gt}{v_T}} - v_T\left(1 - e^{\frac{-gt}{v_T}}\right)$.

Consider some values for t: $t = \frac{v_T}{g}$, $\dot{x} = u_x e^{-1} \approx 0.37u_x$, $\dot{y} = u_y e^{-1} - v_T\left(1 - e^{-1}\right) \approx 0.37u_y - 0.63v_T$

$t = \frac{2v_T}{g}$, $\dot{x} = u_x e^{-2} \approx 0.14u_x$, $\dot{y} = u_y e^{-2} - v_T\left(1 - e^{-2}\right) \approx 0.14u_y - 0.86v_T$

$t = \frac{3v_T}{g}$, $\dot{x} = u_x e^{-3} \approx 0.05u_x$, $\dot{y} = u_y e^{-3} - v_T\left(1 - e^{-3}\right) \approx 0.05u_y - 0.95v_T$

Because $v_T < 0$, as it is downwards, therefore as t increases above $t = \frac{v_T}{g}$, the horizontal velocity decreases exponentially and becomes negligible, so that the particle appears to be falling vertically.

When $t < \frac{v_T}{g}$, the equations of motion with air resistance approximate the equations of motion where air resistance is ignored.

Summary of equations for projectile motion at an angle θ to the horizontal

A particle is projected from the ground with an initial velocity u at an angle θ to the horizontal where $u_x = u\cos\theta$, $u_y = u\sin\theta$. If projected from above the ground, then this initial height of projection needs to be added to the equation for y. k is the constant of proportionality for any resistance.

No resistance

$\ddot{x} = 0$ $\quad$ $\ddot{y} = -g$

$\dot{x} = u\cos\theta$ $\quad$ $\dot{y} = u\sin\theta - gt$

$x = u\cos\theta t$ $\quad$ $y = u\sin\theta t - \frac{1}{2}gt^2$

Resistance proportional to the velocity

$\ddot{x} = -k\dot{x}$ $\quad$ $\ddot{y} = -g - k\dot{y}$

$\dot{x} = u\cos\theta e^{-kt}$ $\quad$ $\dot{y} = \frac{1}{k}\left(\left(g + ku\sin\theta\right)e^{-kt} - g\right)$

$x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$ $\quad$ $y = \frac{\left(g + ku\sin\theta\right)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$

EXERCISE 6.9 PROJECTILES AND RESISTED MOTION

1 An aircraft drops a package of emergency rations to a family stranded in the floods. The aircraft is travelling horizontally at $45.0\,\text{m s}^{-1}$ and is 100 m above the ground. A parachute allows the package to fall with constant speed and hit the ground 10 s after release. Air resistance can be ignored.

(a) Find where the package hits the ground relative to the point from where it was dropped, to the nearest metre.
(b) Find the velocity of the package just before it hits the ground, correct to one decimal place.

2 A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}\,\text{m s}^{-1}$, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection, correct to the nearest tenth of a degree. Air resistance can be ignored.
(b) Find the time of flight of the particle. Give your answer correct to two decimal places.
(c) Find the horizontal distance travelled by the particle. Give your answer correct to one decimal place.
(d) Find the maximum height reached by the particle. Give your answer correct to one decimal place.
(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

3 A cricket ball is thrown from a height of 1 metre with a speed of $30\,\text{m s}^{-1}$ and at an angle of 60° to the horizontal.

(a) Taking the origin at the point of projection and assuming the ground to be level, find the horizontal distance travelled by the ball before it lands. Give your answer correct to one decimal place.
(b) Find the maximum height of the ball above the ground. Give your answer correct to one decimal place.
(c) Find the Cartesian equation of the path of the ball and sketch its path.

4 A baseball is hit horizontally with a speed of $200\,\text{km h}^{-1}$ from a height of 1.225 m. Find how far it travels horizontally before landing, assuming that the baseball field is level. Give your answer correct to one decimal place.

5 A projectile is launched at an angle of 45° to the horizontal. Determine the launching speed, correct to one decimal place, needed to achieve the following (relative to a horizontal plane through the point of projection):

(a) a range of 100 m **(b)** a height of 25 m.

6 A particle is projected from level ground with a speed of $19.6\,\text{m s}^{-1}$ at an angle of projection of α.

(a) Show that the particle reaches its maximum height halfway through its flight.
(b) Find the angle α for which the range of the particle is a maximum and determine this maximum range. Express the maximum range correct to one decimal place.

7 A projectile is projected from the origin with initial speed u and angle of projection α.

(a) Show that the Cartesian equation of its path is $y = x\tan\alpha - \dfrac{gx^2}{2u^2}\sec^2\alpha$.
(b) Hence find an expression for the projectile's range R.
(c) Hence find an expression for the projectile's maximum height y_{max}.
(d) Show that for a given initial speed, the maximum range of the projectile occurs when $\alpha = 45°$ and is given by $R = \dfrac{u^2}{g}$.

8 A golf ball is hit from a point O with initial velocity V at an angle θ to the horizontal ground. The ball first hits the ground at a point P, which is at the same horizontal level as O.

(a) Given that $V\cos\theta = 6u$ and $V\sin\theta = \dfrac{5u}{2}$, find the time taken to travel from O to P, in terms of u and g.
(b) Find the range R of the golf ball in terms of u and g.
(c) Express V in terms of u.
(d) State the golf ball's minimum speed during its flight.

9 The velocity (in m s^{-1}) at time t seconds of a ball hit from a height of 2 metres above ground level is given by $\dot{\underset{\sim}{r}} = 12\underset{\sim}{i} + 9\underset{\sim}{j} + (30 - 9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find the ball's height above ground level, in metres, after 2 seconds.

10 A golf ball is hit from the ground with a velocity $20\underset{\sim}{i}+0\underset{\sim}{j}+15\underset{\sim}{k}\ \text{m s}^{-1}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors horizontally forward, horizontally to the left, and vertically upwards, respectively. After being hit, the ball has a gravitational acceleration of $-10\underset{\sim}{k}\ \text{m s}^{-2}$ and also has a 'hook' (i.e. a horizontal acceleration to the left) of $4\underset{\sim}{j}\ \text{m s}^{-2}$. Air resistance can be ignored.

(a) Find the expression for $\underset{\sim}{r}$, the position vector of the ball at time t.

(b) When the ball hits the ground, how far will it be to the left of the line along the horizontally forward direction?

(c) Find when the speed of the ball is a minimum. Give your answer correct to two decimal places.

(d) Calculate the minimum speed. Give your answer correct to one decimal place.

11 A particle is projected from a point on the horizontal plane with an initial velocity given by the components $u_x = 4\ \text{m s}^{-1}$, $u_y = 6\ \text{m s}^{-1}$. Use $g = 9.8\ \text{m s}^{-2}$.

(a) If the only resistance to the motion is gravity, find the parametric equations of the trajectory of the particle. When does the particle hit the ground?

The particle is projected again into a medium which resists the motion, where the resistance to the motion is directly proportional to the velocity of the particle. Let the constant of proportionality be k.

The parametric equations of the trajectory are given as $x = \dfrac{u_x}{k}\left(1-e^{-kt}\right)$, $y = \dfrac{\left(g+ku_y\right)}{k^2}\left(1-e^{-kt}\right)-\dfrac{gt}{k}$.

(b) Find the parametric equations of the trajectory when **(i)** $k = 0.4$ **(ii)** $k = 0.2$.

(c) Using technology, draw on the same set of axes the graphs of the three trajectories in parts **(a)** and **(b)**.

(d) Discuss what your graphs in part **(c)** tell you about the motion in each case.

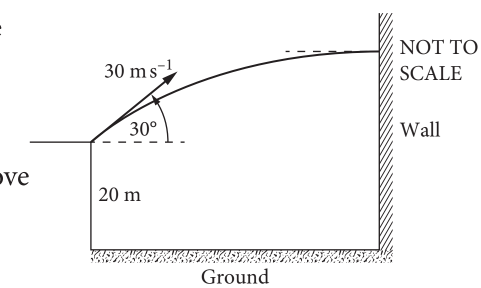
12 The velocity (in m s^{-1}) at time t seconds of a ball thrown at a height of 12 metres above ground level is given by $\dot{\underset{\sim}{r}} = 8\underset{\sim}{i}+3\underset{\sim}{j}+(20-9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find when the ball hits the ground.

13 The trajectory of a projectile fired with speed $u\ \text{m s}^{-1}$ at an angle θ to the horizontal, in a medium whose resistance to the projectile's motion is proportional to the projectile's velocity, is represented by the parametric equations $x = \dfrac{u\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{(10+ku\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$, where k is the constant of proportionality of the resistance.

(a) Show that the greatest height is reached when $t = \dfrac{1}{k}\log_e\left(\dfrac{10+ku\sin\theta}{10}\right)$.

(b) Find the greatest height.

A ball is thrown from a point 20 m above the horizontal ground in the same medium as mentioned above. It is thrown with speed $30\ \text{m s}^{-1}$ at an angle of 30° to the horizontal. At its highest point it hits a wall as shown in the diagram.



(c) If $k = 0.4$, show that the ball hits the wall at a height of 25.8 m above the ground.

(d) What is the horizontal distance of the wall from the point of projection?

14 A projectile is fired from the origin O with an initial velocity $V\ \text{m s}^{-1}$ at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \dfrac{V\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{(10+kV\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$, where $k = 0.2$ is the constant of proportionality of the resistance.

The projectile is fired at an angle of 45° to the horizontal with an initial velocity of $20\sqrt{2}\ \text{m s}^{-1}$.

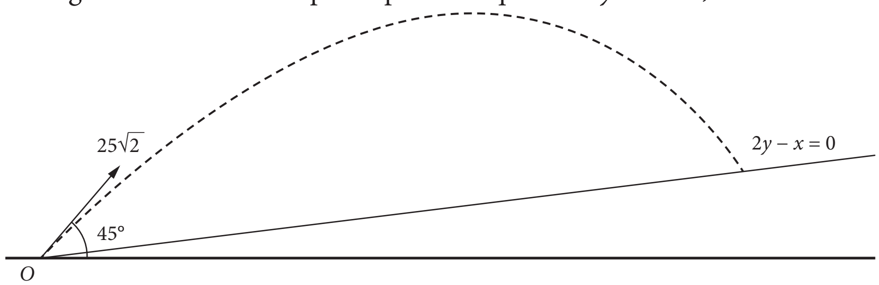
(a) Find when the projectile reaches its greatest height, correct to one decimal place.

(b) Find the greatest height attained, correct to one decimal place.

(c) Show that the projectile hits the ground when $t \approx 3.6$ s **(i)** graphically **(ii)** by substitution.

(d) Find the horizontal range of the projectile.

15 A 100g football is kicked up a slope with equation $2y - x = 0$, as shown.



It is subject to gravity, 10 m s^{-2}, and air resistance, $-0.02\underset{\sim}{v}$ N, where $\underset{\sim}{v}$ is the velocity vector of the football.

The football is launched with speed $25\sqrt{2}$ m s^{-1}, at an angle 45° to the horizontal.

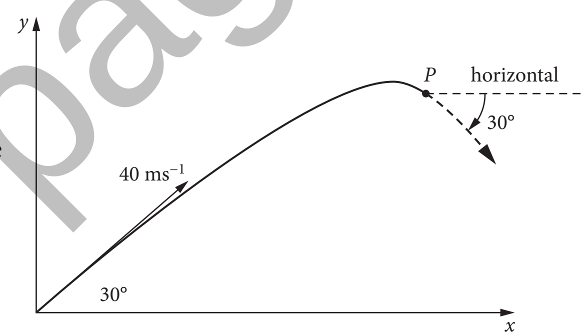
The acceleration of the football is given by $\underset{\sim}{a} = -\underset{\sim}{g} - 0.2\underset{\sim}{v}$, where $\underset{\sim}{g} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$.

(a) Show that the vertical displacement of the football is $y = -50t + 375\left(1 - e^{-0.2t}\right)$.

(b) You are given that the horizontal displacement, in metres, of the football is $x = 125\left(1 - e^{-0.2t}\right)$.

Prove that the Cartesian equation of the trajectory is given by $y = 3x + 250\ \ln\left(1 - \frac{x}{125}\right)$.

16 A particle of mass 1 kg is projected from the origin with speed 40 m s^{-1} at an angle of 30° to the horizontal plane as shown.



The position vector of the particle, at time t seconds after the particle was projected, is given by $\underset{\sim}{r}(t) = \begin{pmatrix} 5\sqrt{3}\left(1 - e^{-4t}\right) \\ \frac{45}{8}\left(1 - e^{-4t}\right) - \frac{5}{2}t \end{pmatrix}$.

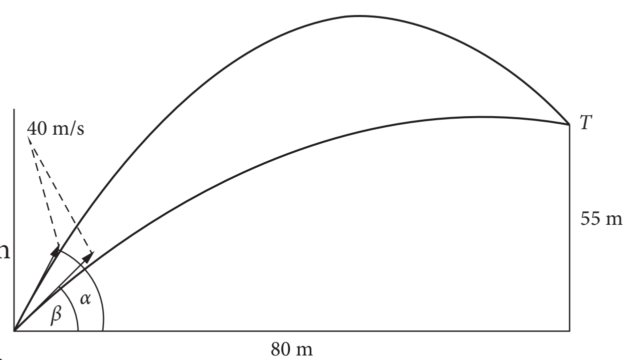
(a) Find the velocity vector, $\underset{\sim}{v}(t)$, of the particle t seconds after it was projected.

(b) At point P the particle is descending at 30° to the horizontal.

Show that the time taken for the particle to reach P is $\ln\sqrt[4]{17}$ seconds.

17 A missile is fired into the ait with a velocity of 40 m s^{-1} at an angle of α to the horizontal.

Shortly afterward, a second missile is fired from the same point with the same velocity, but at a different angle, β, as shown in the diagram.



The missiles both simultaneously hit their target, T, which is at a horizontal distance of 80 m from the point of firing and at a vertical height of 55 m.

(a) Show that the equation of the path of the first missile is given by: $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$.

(b) Show that $\tan\alpha = \frac{5}{2}$ and $\tan\beta = \frac{3}{2}$. Use g = 10 m s^{-2} as an approximation for the acceleration due to gravity.

(c) Determine the time elapsed between the firing of the two missiles.

CHAPTER REVIEW 6

1 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 2$. Its acceleration is given by $\ddot{x} = x + 6$. Using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 3$.

2 A particle moves along the x-axis, starting from a position 2 metres to the right of the origin (i.e. $x = 2$ when $t = 0$), with an initial velocity of $\frac{5\sqrt{2}}{2}$ m s^{-1} and an acceleration $\ddot{x} = x^3 + x$.

(a) Show that $\dot{x} = \frac{x^2 + 1}{\sqrt{2}}$.

(b) Hence find an expression for x in terms of t.

3 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$ where n is a positive constant and x is the displacement of the particle at time t.

(a) Show that the square of the velocity of the particle is $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dx}{dt}$ and a is the amplitude of the motion.

(b) Find the maximum speed of the particle.

(c) Find the maximum acceleration of the particle.

(d) The particle is initially at the origin. Write a formula for x as a function of t. Hence find the first time that the particle's speed is a quarter of its maximum speed.

4 A particle moves with simple harmonic motion on the x-axis about the origin. It is initially at its extreme negative position. The amplitude of the motion is 16 and the particle returns to its initial position every 5 seconds.

(a) Write an equation for the position of the particle at time t seconds.

(b) How much time does the particle take to move from a rest position to the point halfway between the rest position and the equilibrium position?

5 A particle moves in a straight line. Its displacement x metres after t seconds is $x = \sin 2t - \sqrt{3}\cos 2t + 3$.

(a) Prove that the particle is moving in simple harmonic motion about $x = 3$ by showing that $\ddot{x} = -4(x - 3)$.

(b) What is the period of the motion?

(c) Express the velocity of the particle in the form $\dot{x} = A\cos(2t - \alpha)$, where α is in radians.

(d) Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction.

6 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \frac{2t}{9 + t^2}$.

(a) What is the initial velocity of the particle?

(b) Find an expression for the acceleration of the particle.

(c) When is the acceleration zero?

(d) What is the maximum velocity attained by the particle and when does it occur?

(e) Find the position of the particle when $t = 3$.

7 A particle of mass 5 kg moves in a straight line under the action of a force whose magnitude after t seconds is $50 - 10t$ N. Initially the particle is at the origin O with velocity 24 m s^{-1}.

(a) At what time is the particle momentarily at rest?

(b) What is its position at that time?

(c) Describe the motion.

8 An object of mass 10 kg is at rest at the origin. It is acted on by a force that decreases uniformly with the distance travelled by the object, from 50 N at the start to 10 N when the distance travelled is 25 m.

(a) Write the function for this force F in terms of displacement x.

(b) Find the velocity of the object when its displacement is 25 m.

9 A particle of mass 5 kg moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v.

(a) If the resultant force (in newtons) on the particle is $10\sin t$, and $v = 1$ and $x = 1$ when $t = 0$, then find x as a function of t.

(b) If the resultant force (in newtons) on the particle is $15 + 5v$, and $v = 0$ when $t = 0$, then find v as a function of t.

10 A parasailing waterskier is being towed horizontally at a constant speed. The tow rope from the boat makes an angle of 20° above the horizontal and there is tension of 300 N in the tow rope. The waterskier has a mass of 100 kg. A resistance force of 120 N acts against the waterskier in a horizontal direction. A parachute is attached to the skier by a cord that is inclined at an angle α above the horizontal. There is tension of T newtons in the parachute cord. (Use $g = 9.8\ \text{m s}^{-2}$.)

(a) Draw a diagram to show the four forces acting on the waterskier, W.
(b) Explain why the resultant force on the waterskier is zero.
(c) Find T correct to one decimal place and find α correct to the nearest degree.

11 An object is fired vertically from the surface of the Moon with initial velocity v_0 under a gravitational acceleration such that $\ddot{x} = -\frac{k}{x^2}$, where x is the displacement from the centre of the Moon and k is a constant. Let the radius of the Moon be R. The gravitational acceleration at the surface of the Moon is $\frac{g}{6}$.

(a) Find the velocity of the object in terms of its distance x from the centre of the Moon.
(b) Find the value of v_0 for which the object travels a distance of $2R$ from launch before it starts to fall back.
(c) Find the escape velocity.

12 A particle moves from the initial conditions $t = t_0$, $x = x_0$, $v = v_0$ to the final conditions $t = t_F$, $x = x_F$, $v = v_F$ under a force that produces acceleration $\ddot{x}$. Which of the following is incorrect?

A If $\ddot{x} = f(v)$, then $x_F = \int_{v_0}^{v_F} \frac{v}{f(v)}\,dv + x_0$

B If $\ddot{x} = f(v)$, then $x_F = \int_{v_0}^{v_F} \frac{f(v)}{v}\,dv + x_0$

C If $\ddot{x} = f(v)$, then $t_F = \int_{v_0}^{v_F} \frac{dv}{f(v)} + t_0$

D If $\ddot{x} = f(x)$, then $v_F^2 = 2\int_{x_0}^{x_F} f(x)\,dx + v_0^2$

13 **(a)** Show that the range on a horizontal plane of a particle projected upwards at an angle α to the plane and with velocity V metres per second is $\frac{V^2 \sin 2\alpha}{g}$ metres, and that the maximum range is $\frac{V^2}{g}$.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second in a circular pattern. The direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

(b) Prove that the sprinkler, from a fixed position on level ground, will wet the surface of an annular region with centre O and with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively.
(c) Deduce that if the sprinkler is placed appropriately relative to a rectangular garden bed of size 6 m by 3 m, then the entire garden bed may be watered, provided that $\frac{V^2}{2g} \geq 1 + \sqrt{7}$.

14 An underwater camera of mass 0.5 kg is allowed to fall vertically from the ocean surface into a deep ocean trench. As it falls to the ocean floor, it is acted upon by gravity and by a resistance of $2v$ newtons, where $v\ \text{m s}^{-1}$ is the velocity of the camera t seconds after beginning its descent.

(a) Show that the equation of motion of the camera is $\ddot{x} = g - 4v$.
(b) Find v as a function of t.
(c) Find the terminal velocity of the camera.
(d) Find the time taken for the camera to reach half of its terminal velocity.
(e) It takes 50 seconds for the camera to reach the ocean floor. Find the depth of the ocean at that point.

15 A particle moves so that its position vector $\underset{\sim}{r}$ at time t is given by $\underset{\sim}{r} = 3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}$, $t \geq 0$.

(a) Show that the particle moves in a circle and find the Cartesian equation of its path.
(b) Show that the particle moves with constant speed.
(c) Show that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

16 The position vector of a particle at time t seconds, $t \geq 0$, is $\underset{\sim}{r} = (1+\sin 4t)\underset{\sim}{i} + (2-\cos 4t)\underset{\sim}{j}$ metres.

(a) Show that the particle moves in a circle and sketch its path.
(b) Show that the particle's acceleration is always perpendicular to its velocity.

17 The position vector of a particle at time t, $t \geq 0$, is $\underset{\sim}{r} = 2\cos 3t\,\underset{\sim}{i} + 2\sin 3t\,\underset{\sim}{j} + 3t\,\underset{\sim}{k}$.

Show that the magnitudes of the particle's velocity and its acceleration are constant.

18 A particle moves so that its position vector at time t is given by $\underset{\sim}{r} = 3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}$, $0 \leq t \leq 2\pi$.

(a) Find the Cartesian equation of the path of the particle and sketch the path.
(b) Find when the velocity of the particle is perpendicular to its position vector and hence find the position vectors at these times.
(c) Sketch the graph of the speed function and find the maximum and minimum speeds of the particle.
(d) Show that the particle's acceleration is directed towards the origin and is equal in magnitude to the particle's distance from the origin.
(e) Find when the acceleration is perpendicular to the velocity.

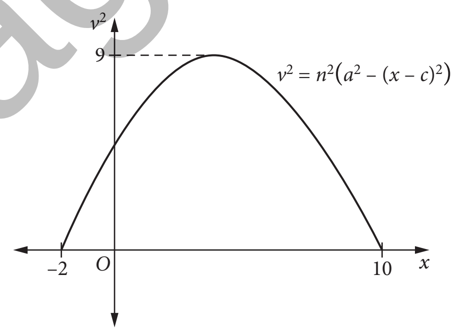
19 Which of the following functions does NOT describe simple harmonic motion?

A $x = \sin^2 3t - \cos^2 3t$ **B** $x = 5\cos\left(2t + \frac{\pi}{6}\right)$

C $x = 2\sin 4t - 4\cos 2t$ **D** $x = 2\sin^2 2t - 5\cos 4t + 2$

20 A particle is moving along the x-axis in simple harmonic motion.
The displacement of the particle is x metres and its velocity is v m s^{-1}.
The graph of $v^2 = n^2\left(a^2 - (x-c)^2\right)$, where a, c and n are positive constants is shown.

What are the values of a, c and n?



A $a = 4, c = 6, n = \frac{1}{2}$ **B** $a = 4, c = 6, n = \frac{3}{2}$

C $a = 6, c = 4, n = \frac{1}{2}$ **D** $a = 6, c = 4, n = \frac{3}{2}$

21 A crate of mass 15 kg is being pulled along a smooth horizontal surface by a rope inclined at 30° to the horizontal. $\left(g = 10\,\text{m s}^{-2}\right)$.
The tension in the rope is 18 N.

(a) Draw a force diagram for this situation.
(b) Find the acceleration of the box.
(c) Find the normal force between the box and the floor.