CHAPTER 1
Algebraic techniques

1.1 SIMPLIFYING ALGEBRAIC EXPRESSIONS

When adding and subtracting algebraic expressions, you can only combine like terms (that is, algebraic parts that have the same pronumerals). Be careful that when subtracting terms with brackets, the subtraction must be applied as a negative to each term inside the brackets (as in part **(c)** of the example below).

Example 1

Simplify each expression by collecting the like terms.

(a) $3x + 2y + 5x - 6y$ (b) $x^2 + 2x - x + 3x^2$ (c) $2(3a - 4b) - 3(a - 5b)$

Solution

(a) $3x + 2y + 5x - 6y$
$= 3x + 5x + 2y - 6y$
$= 8x - 4y$

(b) $x^2 + 2x - x + 3x^2$
$= x^2 + 3x^2 + 2x - x$
$= 4x^2 + x$

(c) $2(3a - 4b) - 3(a - 5b)$
$= 6a - 8b - 3a + 15b$
$= 6a - 3a - 8b + 15b$
$= 3a + 7b$

EXERCISE 1.1 SIMPLIFYING ALGEBRAIC EXPRESSIONS

Simplify each expression by collecting the like terms.

1 $3x + 5 + 7x + 10$ **2** $7x - 3 + 3x - 2$ **3** $4a + b - a - 4b$

4 $6ab + 3ab + 5a + 4a$ **5** $3xy + 2xy - yx$ **6** $3a^2b - 3ab^2 + 2a^2b$

7 $2x^2y + 3x^2y^2 - x^2y + 3x^2y^2$ **8** $3abc + 5bca - 2cba$ **9** $12mn + 3m - 6mn - m$

10 $x^2 - 3x + 2x + 4x^2$ **11** $2x^2 + 5y^2 - 4x^2$

Simplify each expression by expanding the brackets and then collecting the like terms.

12 $5a - 3(a + b)$ **13** $4(2x - y) - 6x$ **14** $8m - 5(2m - 3n)$

15 $3(2x + 5y) + 4(x - y)$ **16** $5(2x + 3) - 5(x + 7)$ **17** $6(2a + 3b) + 3(a - b)$

18 $5a(a + 2) - 3a(a + 1)$ **19** $5x(x - 2y) + 3x(2x - y)$ **20** $2a + 3b - (a - b)$

21 $x + 5y - (3x + 2y)$ **22** $5x(2x + 1) - (x^2 + x)$ **23** $15(x - 2) + 4(3x - 3)$

24 $3x(x - 2) - 4(x - 1)$ **25** $3(x^2 + 5x - 1) - (2x^2 + x - 2)$ **26** $5x + 2y - 3 - (x - 7y + 9)$

27 The expression $a(a + 1) - 3(2a + 1)$ simplifies to:

A $a^2 - 7a - 3$ B $a^2 - 5a + 3$ C $a^2 - 5a - 3$ D $a^2 - 7a + 3$

28 The expression $3(m^2 - m) - 2(m^2 + 2m + 5)$ simplifies to:

A $5m^2 - 7m - 10$ B $m^2 - 7m - 10$ C $m^2 + m - 10$ D $m^2 - 7m + 10$

1.2 SUBSTITUTION IN FORMULAE

Example 2

If $V = \pi r^2 h$, find:

(a) V when $r = 3.5$, $h = 5$

(b) r when $V = 275$, $h = 14$

Solution

(a) $V = \pi \times 3.5^2 \times 5$

$= 61.25\pi$ (exact value)

$= 192.4$ correct to one decimal place

(b) $r^2 = \dfrac{V}{\pi h}$

$r^2 = \dfrac{275}{14 \times \pi}$

$r = \sqrt{\dfrac{275}{14\pi}}$ (exact value)

$r = 2.5$ correct to one decimal place

EXERCISE 1.2 SUBSTITUTION IN FORMULAE

Use the value of π on your calculator. Give your answer correct to one decimal place when needed.

1 If $P = 2(l + b)$, find the value of P when $l = 20$, $b = 12$.

2 If $E = IR$, find E when $I = 2.4$, $R = 40$.

3 If $F = ma$, find F when $m = 50$, $a = 0.2$.

4 If $F = \dfrac{9C}{5} + 32$, find: (a) F when $C = 60$ (b) C when $F = 41$.

5 If $A = \pi r^2$, find A when $r = 3.5$.

6 If $V = \pi r^2 h$, find V when $r = 4.2$, $h = 10$.

7 If $E = mc^2$, find: (a) E when $m = 10$, $c = 1.6$ (b) c when $E = 13.5$, $m = 1.5$.

8 If $v = u + at$, find v when $u = 20$, $a = 1.8$, $t = 10$.

9 If $s = ut + \dfrac{1}{2}at^2$, find: (a) s when $u = 5$, $a = 6$, $t = 2.4$ (b) a when $s = 50$, $t = 2.5$, $u = 10$.

10 If $v^2 = u^2 + 2as$, find v when $u = 12$, $a = 2$, $s = 20.25$.

11 If $s = \dfrac{1}{2}(u + v)t$, find s when $u = 2.6$, $v = 3.2$, $t = 2.5$.

12 If $S = 2\pi rh$, find S when $r = 2.5$, $h = 3.5$.

13 If $r = \sqrt{\dfrac{A}{\pi}}$, find: (a) r when $A = 154$ (b) A when $r = 1.75$.

14 If $E = \dfrac{m}{2}\left(v^2 - u^2\right)$, find E when $m = 4$, $v = 4$, $u = 2$.

15 If $t = a + (n - 1)d$, find: (a) t when $a = 3.8$, $n = 20$, $d = 0.2$ (b) n when $a = 5.6$, $d = 5$, $t = 25.6$.

16 If $F = \dfrac{m(v - u)}{t}$, find F when $m = 20$, $v = 4$, $u = 2$, $t = 6$.

17 If $t = ar^5$, find t when $a = 64$, $r = 0.5$.

18 If $S = \dfrac{a\left(r^3 - 1\right)}{r - 1}$, find S when $a = 5$, $r = 3$.

19 If $A = \pi(R^2 - r^2)$, find A when $R = 5.6$, $r = 1.4$.

20 If $V = \pi(R^2 - r^2)h$, find V when $R = 0.9$, $r = 0.2$, $h = 1.5$.

21 If $V = \frac{1}{3}\pi r^2 h$, find V when $r = 3$, $h = 3.5$.

22 If $P = \sqrt{\frac{2R - V}{5}}$, find: **(a)** P when $R = 50$, $V = 20$ **(b)** V when $P = 0.2$, $R = 20$.

23 If $W = \frac{1}{3}d\pi r^2 h$, find W when $d = 3$, $r = \frac{7}{11}$, $h = \frac{11}{14}$.

24 If $f = \frac{vu}{v + u}$, find: **(a)** f when $v = 20$, $u = 25$ **(b)** v when $f = 20$, $u = 25$.

25 If $A = P\left(1 + \frac{r}{100}\right)^n$, find A when $P = 1000$, $r = 10$, $n = 2$.

1.3 BASIC POLYNOMIALS

Common terms

A **monomial** is an expression that contains only **one** term, e.g. $5x$, x^2, $2ab$, $5a^2b^3$.

A **binomial** is an expression that contains **two** terms added or subtracted, e.g. $x + y$, $3a - 2b$, $x^2 + 1$, $3y - 4$.

A **trinomial** is an expression that contains **three** terms added or subtracted, e.g. $x^2 - 5x + 6$, $x + y - 4$, $4x^2 - 2xy + y^2$, $m + n - p$.

A **quadratic trinomial** is a trinomial of the form $ax^2 + bx + c$ (where $a \neq 0$, $b \neq 0$, $c \neq 0$); a is the coefficient of x^2, b is the coefficient of x, and c is the constant term.

Standard results

$$(x + m)(x + n) = x^2 + (m + n)x + mn$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$

In each of these results, the expression on the left-hand side has been **expanded** to obtain the expression on the right.

If we start with the expression on the right-hand side, then we can **factorise** it to obtain the (usually) shorter form on the left.

Example 3

Expand and simplify each expression.

(a) $(x + 2)(x + 3)$ **(b)** $(3x - 2)(2x + 3)$ **(c)** $(2y + 5)^2$

(d) $(3x - 4)(3x + 4)$ **(e)** $(x + 2)(x^2 - 5x + 6)$ **(f)** $(x - 1)(x + 2)(x + 3)$

Solution

(a) $(x + 2)(x + 3)$
$= x(x + 3) + 2(x + 3)$
$= x^2 + 3x + 2x + 6$
$= x^2 + 5x + 6$

(b) $(3x - 2)(2x + 3)$
$= 6x^2 + 9x - 4x - 6$
$= 6x^2 + 5x - 6$

(c) $(2y + 5)^2$
$= 4y^2 + 20y + 25$

(d) $(3x - 4)(3x + 4)$
$= 9x^2 - 16$

(e) $(x + 2)(x^2 - 5x + 6)$
$= x(x^2 - 5x + 6) + 2(x^2 - 5x + 6)$
$= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12$
$= x^3 - 3x^2 - 4x + 12$

(f) $(x - 1)(x + 2)(x + 3)$
$= (x - 1)(x^2 + 5x + 6)$
$= x^3 + 5x^2 + 6x - x^2 - 5x - 6$
$= x^3 + 4x^2 + x - 6$

EXERCISE 1.3 BASIC POLYNOMIALS

Write the expansion of the following.

1 $(x+5)(x+1)$

2 $(x-2)(x-3)$

3 $(a-3)(a+4)$

4 $(x-2)^2$

5 $(y+7)^2$

6 $(2x+3)(x+5)$

7 $(3x-4)(x-2)$

8 $(3m+7)(2m-1)$

9 $(3x+2)(3x+2)$

10 $(2p-9)(2p+9)$

11 $(3x+2)(2x+3)$

12 $(4p-5)^2$

13 $(3x+4)^2$

14 $(x-3)(2x^2+3x+1)$

15 $(3x^2-5x+2)(2x-4)$

16 $x(x-2)(x+2)$

17 $(x-1)(x-1)(x-2)$

18 $2(x-1)(x-2)(x-3)$

19 $(x^2+5)(x^2-2x-3)$

20 $(x-2)(x+2)(x+2)$

21 $(x^2-y^2)^3$

22 The correct expansion of $\left(\sqrt{x}+\sqrt{y}\right)^2$ is:

A $x^2+2xy+y^2$ B $x+y+2xy$ C $x+y+2\sqrt{xy}$ D $x^2+y^2+2\sqrt{xy}$

23 Indicate whether each answer is a correct or incorrect factorisation of $x^2-4xy+4y^2$.

(a) $(x+2y)^2$ (b) $(2y-x)^2$ (c) $(2x-y)^2$ (d) $(x-2y)^2$

1.4 FACTORISING BY GROUPING IN PAIRS

This method is used when there are four terms in the expression.

Example 4

Factorise:

(a) $bx+by+cx+cy$ (b) $m^2-mn-2m+2n$

Solution

Group in pairs: (a) $\underbrace{bx+by}+\underbrace{cx+cy}$ (b) $\underbrace{m^2-mn}-\underbrace{2m+2n}$

Take out common factor: $=b(x+y)+c(x+y)$ $=m(m-n)-2(m-n)$

Take out common factor: $=(x+y)(b+c)$ $=(m-2)(m-n)$

EXERCISE 1.4 FACTORISING BY GROUPING IN PAIRS

Factorise:

1 $a(x+2)+b(x+2)$

2 $3a(2b-3c)-m(2b-3c)$

3 $p(a+b)+q(a+b)-r(a+b)$

4 $x^2(2x-1)+4(2x-1)$

5 $ax+4a+bx+4b$

6 $x^2-xy+xz-yz$

7 $2xy+2xz+y+z$

8 $a^2-ab-ac+bc$

9 $10y-25y^2+4x-10xy$

10 $a^3+3a^2b+ab^2+3b^3$

11 $ac-2bc-2ad+4bd$

12 $3xy-6y+7x-14$

13 $x^2-2xy-xz+2yz$

14 $a^3-a^2b-ab+b^2$

15 $2mn+2mp+pn^2+p^2n$

16 $x^3+3x^2+4x+12$

17 $p^2q-pq^2+5p-5q$

18 m^2p+m^2+np+n

19 $x^2y + x^2 + y + 1$

20 $ab - 3a - 4b + 12$

21 $2x - 6y - xy + 3y^2$

22 When $3m^2 - 3mn - m + n$ is factorised, the answer is:

A $(3m-1)(m-n)$ B $(3m-n)(m-1)$ C $(3m-1)(m+n)$ D $(3m+1)(m-n)$

23 Indicate whether each answer is a correct or incorrect factorisation of $2x^3 - 2x^2 - 2x + 2$.

(a) $2(x+1)(x+1)(x-1)$ (b) $2(x+1)(x-1)^2$ (c) $2(x+1)(x-1)(x-1)$ (d) $2(x-1)(x+1)^2$

1.5 STANDARD FACTORISATIONS

Factorising using the difference of two squares

Remember the difference of two squares: $a^2 - b^2 = (a-b)(a+b)$

Example 5

Factorise:

(a) $a^2 - 25$ (b) $9x^2 - 49$ (c) $(x+1)^2 - (y-1)^2$ (d) $a^3 - a^2b - ab^2 + b^3$

Solution

(a) $a^2 - 25 = a^2 - 5^2$
$= (a-5)(a+5)$

(b) $9x^2 - 49 = (3x)^2 - 7^2$
$= (3x-7)(3x+7)$

(c) $(x+1)^2 - (y-1)^2$
$= [(x+1) - (y-1)][(x+1) + (y-1)]$
$= (x - y + 2)(x + y)$

(d) $a^3 - a^2b - ab^2 + b^3$
$= a^2(a-b) - b^2(a-b)$
$= (a^2 - b^2)(a-b)$
$= (a-b)(a+b)(a-b)$
$= (a-b)^2(a+b)$

Sum and difference of two cubes

Two important identities are: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

The identities can be verified by expanding the right-hand side.

Example 6

Factorise:

(a) $x^3 - 8$ (b) $27y^3 + 64x^3$ (c) $(x+2)^3 + y^3$ (d) $x^2y^3 - z^2y^3 - x^2w^3 + z^2w^3$

Solution

(a) $x^3 - 8 = x^3 - 2^3$
$= (x-2)(x^2 + 2x + 4)$

(b) $27y^3 + 64x^3 = (3y)^3 + (4x)^3$
$= (3y + 4x)(9y^2 - 12xy + 16x^2)$

(c) $(x+2)^3 + y^3$
$= (x + 2 + y)[(x+2)^2 - (x+2)y + y^2]$
$= (x + 2 + y)(x^2 + 4x + 4 - xy - 2y + y^2)$

(d) $x^2y^3 - z^2y^3 - x^2w^3 + z^2w^3$
$= y^3(x^2 - z^2) - w^3(x^2 - z^2)$
$= (x^2 - z^2)(y^3 - w^3)$
$= (x-z)(x+z)(y-w)(y^2 + yw + w^2)$

EXERCISE 1.5 STANDARD FACTORISATIONS

Factorise:

1 $m^2 - 1$ **2** $x^2 - 16$ **3** $64 - m^2$ **4** $9a^2 - 25$

5 $x^2 - 0.36$ **6** $a^2b^2 - c^2$ **7** $9x^2 - 4y^2$ **8** $(x+1)^2 - 9$

9 $x^2 - y^2z^2$ **10** $\frac{a^2}{25} - 1$ **11** $p^2 - \frac{1}{4}$ **12** $\frac{x^2}{4} - \frac{1}{9}$

13 $(a+2)^2 - 4$ **14** $x^2 - (y+z)^2$ **15** $99^2 - 1$ **16** $523^2 - 477^2$

17 $a^3b - ab^3$ **18** $12a^3 - 3ab^2$ **19** $3x^2y - 27y$ **20** $(x+y)^2 - 4$

21 $a^2 - (a-b)^2$ **22** $x^3 - x^2y - 9x + 9y$ **23** $x^3 + 3x^2 - 4x - 12$ **24** $p^2q - p^2 - 16q + 16$

25 $a^2x - x$ **26** $48a^2 - 75b^2$ **27** $(1+h)^2 - 1$ **28** $\frac{x^2}{25} - y^2$

29 When $(p+2)^2 - (p-2)^2$ is factorised, the answer is:

A $2p^2 + 8$ **B** $-8p$ **C** $2p^2 - 8$ **D** $8p$

30 Indicate whether each answer is a correct or incorrect factorisation of $\frac{a^2}{b^2} - \frac{b^2}{a^2}$.

(a) $\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{b} + \frac{b}{a}\right)$ **(b)** $\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{b}{a} + \frac{a}{b}\right)$ **(c)** $\left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{b} - \frac{b}{a}\right)$ **(d)** $\frac{(a-b)(a+b)(a^2+b^2)}{a^2b^2}$

31 $y^3 - 125$ **32** $z^3 + 1$ **33** $8p^3 + 27$ **34** $216 - a^3$

35 $(x+5)^3 + (x-2)^3$ **36** $(2x+3)^3 - (x-4)^3$ **37** $b^6 - a^6$ **38** $64a^3 + 8b^3$

39 $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$ **40** $p^7x^4 - p^4x^7$ **41** $x^6 + y^6$ **42** $\frac{8}{a^3} - \frac{27}{b^3}$

43 $a^3m^3 + a^3n^3 - b^3n^3 - b^3m^3$ **44** $4x^5 - 9x^3 - 4x^2 + 9$ **45** $(x+h)^3 - x^3$

46 $a^3 + (a-b)^3$ **47** $(a+b)^3 - (a-b)^3$ **48** $(2x+1)^3 - (2x-1)^3$

49 $8 - (2-x)^3$ **50** $a^5b^4 - a^2b$ **51** $2(x-y)^3 + 54$

52 When $1000p^3 - q^6$ is factorised, the answer is:

A $(10p - q)(100p^2 + 10pq + q^2)$ **B** $(10p - q^2)(100p^2 + 10pq^2 + q^4)$

C $(10p + q)(100p^2 - 10pq + q^2)$ **D** $(10p + q^2)(100p^2 - 10pq^2 + q^4)$

53 When $(2x+1)^3 + (2x-1)^3$ is factorised, the answer is:

A $2(12x^2 + 1)$ **B** $4x(12x^2 + 1)$ **C** $2(4x^2 + 3)$ **D** $4x(4x^2 + 3)$

1.6 FACTORISING QUADRATIC TRINOMIALS

To factorise quadratic trinomials, you must remember how to expand binomial products and then work backwards. We know the following:

- $(x+m)(x+n) = x^2 + (m+n)x + mn = x^2 +$ (sum of m and n)x + (the product of m and n)
- $(x-m)(x-n) = x^2 - (m+n)x + mn = x^2 +$ (sum of $-m$ and $-n$)x + (the product of $-m$ and $-n$)
- $(x+m)(x-n) = x^2 + (m-n)x - mn = x^2 +$ (sum of m and $-n$)x + (the product of m and $-n$)

To factorise $x^2 + 5x + 6$ you must write it in the form $(x+m)(x+n)$, where $m + n = 5$ and $mn = 6$. This means you must find two numbers whose sum is 5 and whose product is 6.

Example 7

Factorise:

(a) $x^2 + 5x + 6$ (b) $x^2 - 7x + 10$ (c) $x^2 + x - 12$ (d) $x^2 - 6x + 9$ (e) $x^2 - 5x - 24$

Solution

(a) $x^2 + 5x + 6$

Write: $x^2 + 5x + 6 = (x + m)(x + n)$

Look for numbers m and n whose sum is 5 and whose product is 6.

List possible factors of 6 and check the sum: $6 \times 1 = 6$ $\quad 6 + 1 = 7$

$3 \times 2 = 6$ $\quad 3 + 2 = 5$

Hence: $x^2 + 5x + 6 = (x + 3)(x + 2)$

The information could also be set out using the cross method:

x ╳ ~~6~~ 3
x ~~1~~ 2

The correct pair will give $5x$ when multiplied across.

(b) $x^2 - 7x + 10$

Write: $x^2 - 7x + 10 = (x + m)(x + n)$

Look for numbers m and n whose sum is -7 and whose product is 10.

Since the sum is negative and the product is positive, both the numbers are negative.

List possible factors of 10 and check the sum: $-10 \times (-1) = 10$ $\quad -10 + (-1) = -11$

$-5 \times (-2) = 10$ $\quad -5 + (-2) = -7$

Hence: $x^2 - 7x + 10 = (x - 5)(x - 2)$

(c) $x^2 + x - 12$

Write: $x^2 + x - 12 = (x + m)(x + n)$

Look for numbers m and n whose sum is 1 and whose product is -12.

Since the sum is positive and the product is negative, the numbers have different signs and the larger number is positive.

List possible factors of -12 and check the sum: $12 \times (-1) = -12$ $\quad 12 + (-1) = 11$

$6 \times (-2) = -12$ $\quad 6 + (-2) = 4$

$4 \times (-3) = -12$ $\quad 4 + (-3) = 1$

Hence: $x^2 + x - 12 = (x + 4)(x - 3)$

Using the cross method:

x ╳ ~~12~~ ~~6~~ 4
x ~~−1~~ ~~−2~~ −3

The correct pair will give x when multiplied across.

(d) $x^2 - 6x + 9$

Write: $x^2 - 6x + 9 = (x + m)(x + n)$

Look for numbers m and n whose sum is -6 and whose product is 9.

Since the sum is negative and the product is positive, both the numbers are negative.

List possible factors of 9 and check the sum: $-9 \times (-1) = 9$ $\quad -9 + (-1) = -10$

$-3 \times (-3) = 9$ $\quad -3 + (-3) = -6$

Hence: $x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

(e) $x^2 - 5x - 24$

Write: $x^2 - 5x - 24 = (x + m)(x + n)$

Look for numbers m and n whose sum is -5 and whose product is -24.

Since the sum is negative and the product is negative, the numbers have different signs and the smaller number is positive.

List possible factors of -24 and check the sum:

$-24 \times 1 = -24$	$-24 + 1 = -23$
$-12 \times 2 = -24$	$-12 + 2 = -10$
$-6 \times 4 = -24$	$-6 + 4 = -2$
$-8 \times 3 = -24$	$-8 + 3 = -5$

Hence: $x^2 - 5x - 24 = (x - 8)(x + 3)$

x	╳	~~-24~~	~~-12~~	~~-6~~	-8
x		~~1~~	~~2~~	~~4~~	3

The correct pair will give $-5x$ when multiplied across.

With practice, you will be able to write the factors simply by looking at the sum and product.

MAKING CONNECTIONS

Factorising quadratic trinomials

Use technology to check the factorisation of quadratic trinomials.

EXERCISE 1.6 FACTORISING QUADRATIC TRINOMIALS

Factorise the following quadratic trinomials:

1 $x^2 + 4x + 3$
2 $x^2 + 10x + 21$
3 $x^2 + 11x + 24$
4 $a^2 + 12a + 32$
5 $m^2 + 9m + 20$
6 $x^2 + 13x + 12$
7 $x^2 + 8x + 12$
8 $x^2 - 7x + 12$
9 $x^2 - 13x + 12$
10 $x^2 - 8x + 12$
11 $p^2 + 2p - 15$
12 $p^2 + 14p - 15$
13 $p^2 - 2p - 15$
14 $p^2 - 14p - 15$
15 $x^2 - 2x - 35$
16 $x^2 - 3x - 10$
17 $x^2 + 17x + 72$
18 $a^2 - 4a - 12$
19 $x^2 - 7x + 6$
20 $x^2 - x - 72$
21 $x^2 + 6x - 72$
22 $x^2 - 21x - 72$
23 $a^2 + 13a + 30$
24 $x^2 - x - 42$
25 $x^2 - 19x - 42$
26 $x^2 + 19x - 42$

27 The factors of $x^2 - 11x - 42$ are:

A $(x + 14)(x - 3)$ **B** $(x - 7)(x + 6)$ **C** $(x - 6)(x + 7)$ **D** $(x - 14)(x + 3)$

28 Indicate whether each answer is a correct or incorrect factorisation of $x^2 - 8x + 7$.

(a) $(x + 1)(x - 7)$ **(b)** $(1 - x)(7 - x)$ **(c)** $(x - 1)(x + 7)$ **(d)** $(x - 1)(x - 7)$

1.7 FACTORISING NON-MONIC TRINOMIALS

When the x^2 term in the quadratic has a coefficient other than 1, finding the factors becomes more difficult because there are more possibilities. You must find factors of the coefficient of x^2 as well as the factors of the constant term and get the correct pairs together. You could use trial and error, but the cross method is easier because it keeps the information more organised.

Example 8

Factorise $6x^2 + 19x + 10$.

Solution

Write: $6x^2 + 19x + 10 = (ax + m)(bx + n) = abx^2 + (an + bm)x + mn$

This gives: $ab = 6$, $an + bm = 19$, $mn = 10$

List the factors of 6: 6, 1 or 3, 2

List the factors of 10: 10, 1 or 5, 2

List possible binomial factors:

$(6x + 10)(x + 1)$	$(6x + 1)(x + 10)$	$(6x + 5)(x + 2)$	$(6x + 2)(x + 5)$
$(3x + 10)(2x + 1)$	$(3x + 1)(2x + 10)$	$(3x + 5)(2x + 2)$	$(3x + 2)(2x + 5)$

You can expand the binomial factors to see which answer gives the original quadratic trinomial.

Before doing this, you can eliminate any possibility that has a common factor, because there is no common factor in the original quadratic trinomial. This means you can eliminate the answers containing $(6x + 10)$, $(6x + 2)$, $(2x + 10)$ and $(2x + 2)$, because they all have a common factor of 2 (and the original quadratic trinomial does not).

Expand the others:

$(6x + 1)(x + 10) = 6x^2 + 61x + 10$ Not correct

$(6x + 5)(x + 2) = 6x^2 + 17x + 10$ Not correct

$(3x + 10)(2x + 1) = 6x^2 + 23x + 10$ Not correct

$(3x + 2)(2x + 5) = 6x^2 + 19x + 10$ Correct

Hence: $6x^2 + 19x + 10 = (3x + 2)(2x + 5)$

Alternatively, using the cross method:

$$\begin{matrix} 6x & \times & 10 & 1 & 5 & 2 \\ x & & 1 & 10 & 2 & 5 \end{matrix} \quad \text{or} \quad \begin{matrix} 3x & \times & 10 & 1 & 5 & 2 \\ 2x & & 1 & 10 & 2 & 5 \end{matrix}$$

The correct pair will give $19x$ when multiplied across.

Hence: $6x^2 + 19x + 10 = (3x + 2)(2x + 5)$

Example 9

Factorise $4x^2 - 4x - 15$.

Solution

The factors of $4x^2$ are either $4x$ and x, or $2x$ and $2x$.

The factors of -15 are -15 and 1, 15 and -1, -5 and 3, or 5 and -3.

Set up the cross method:

$$\begin{matrix} 4x & \times & -15 & 15 & -5 & 5 \\ x & & 1 & -1 & 3 & -3 \end{matrix} \quad \text{or} \quad \begin{matrix} 4x & \times & 1 & -1 & 3 & -3 \\ x & & -15 & 15 & -5 & 5 \end{matrix}$$

or

$$\begin{matrix} 2x & \times & -15 & 15 & -5 & 5 \\ 2x & & 1 & -1 & 3 & -3 \end{matrix} \quad \text{or} \quad \begin{matrix} 2x & \times & 1 & -1 & 3 & -3 \\ 2x & & -15 & 15 & -5 & 5 \end{matrix}$$

The only combination that gives $-4x$ is:

$$\begin{matrix} 2x & \times & -5 \\ 2x & & 3 \end{matrix}$$

Hence: $4x^2 - 4x - 15 = (2x - 5)(2x + 3)$

Example 10

Factorise $3x^2 + 8x - 16$.

Solution

The factors of $3x^2$ are $3x$ and x.

The factors of -16 are -16 and 1; 16 and -1; -8 and 2; 8 and -2; -4 and 4; or 4 and -4.

Set up the cross method:

$3x$	╳	-16	1	16	-1	-8	2	-2	8	-4	4
x		1	-16	-1	16	2	-8	8	-2	4	-4

The only combination that gives $8x$ is:

$3x$	╳	-4
x		4

Hence: $3x^2 + 8x - 16 = (3x - 4)(x + 4)$

EXERCISE 1.7 FACTORISING NON-MONIC TRINOMIALS

Factorise:

1 $2x^2 + 3x + 1$ **2** $3x^2 + 11x - 4$ **3** $2x^2 + 7x + 6$ **4** $4a^2 + 13a + 3$

5 $3a^2 - 5a + 2$ **6** $8x^2 - 14x + 3$ **7** $13c^2 - 7c - 6$ **8** $8x^2 + 14x + 5$

9 $3x^2 - 17x + 10$ **10** $6a^2 - 13a - 63$ **11** $3x^2 - 11x - 4$ **12** $10x^2 - 11x - 8$

13 $2x^2 + 3x - 2$ **14** $4x^2 - 12x + 9$ **15** $9x^2 - 12x + 4$ **16** $2x^2 - 9x + 10$

17 $6x^2 - 85x + 14$ **18** $y^2 - 2y - 3$ **19** $12y^2 + 14y - 6$ **20** $6x^2 - 25x + 14$

21 $6x^2 - 29x + 28$ **22** $6x^2 - 19x + 14$ **23** $6x^2 - 20x + 14$ **24** $8x^2 + 2x - 3$

25 $6p^2 + 25p + 21$ **26** $10a^2 - 11a - 6$ **27** $12y^2 + 28y - 5$ **28** $24x^2 - 59x + 36$

29 $15x^2 - 19x + 6$ **30** $3x^2 - 2x - 1$ **31** $9x^2 + 9x - 10$ **32** $2x^2 - 9x + 4$

33 The factors of $8x^2 - 6x - 9$ are:

A $(4x - 3)(2x + 3)$ **B** $(8x - 9)(x + 1)$ **C** $(4x + 3)(2x - 3)$ **D** $(4x - 9)(2x + 1)$

34 Indicate whether each answer is a correct or incorrect factorisation of $4x^2 + 12x + 9$.

(a) $(2x + 3)(2x + 3)$ **(b)** $(3 + 2x)(3 + 2x)$ **(c)** $(2x + 3)^2$ **(d)** $(2x - 3)^2$

1.8 MIXED FACTORISATIONS

To factorise the expressions in the next exercises, you will need to use **one or more** of the techniques learned so far:

- remove a common factor
- group in pairs and remove common factors
- difference of two squares
- quadratic trinomials.

EXERCISE 1.8 MIXED FACTORISATIONS

Factorise completely:

1 $x^2 - 3x$ **2** $2a^3 - 8a$ **3** $x^2 - 9$ **4** $x^2 - 8x - 9$

5 $3x^2y - 12y^3$ **6** $5x^3y - 20xy^3$ **7** $1 - (b + c)^2$ **8** $10x^2 + 9x - 1$

9 $(a+b)^2 - b^2$ **10** $6x^2 - 24$ **11** $a^2 - a - 42$ **12** $a(m+n) - b(m+n)$

13 $2x^3 + 14x^2 - 16x$ **14** $3a^3 + 24a^2 + 21a$ **15** $(x+2y)^2 - 4$ **16** $ab^2 + abc + abd$

17 $x^2 - 36y^2$ **18** $x(y-z) + y(y-z)$ **19** $4x^2 - 28x - 480$ **20** $bx^2 - 14bxy + 49by^2$

21 $6y^3 + 3y^2 - 3y$ **22** $6y^3 + 26y^2 + 8y$ **23** $15a^2 - 60$ **24** $9mn - 25m^3n^3$

25 $5a^2x - 125x$ **26** $(x+y)^2 - (x-y)^2$ **27** $5t^3 + 5t^2 - 360t$ **28** $m^2 - mn + 6m - 6n$

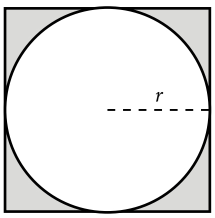
29 $x^2(x+3) - 4(x+3)$ **30** $mx^2 - xy + ly - mlx$

31 The factors of $4 - (x+1)^2$ are:

A $(3-x)(5+x)$ **B** $(2-x)(2+x)$ **C** $(1+x)(3-x)$ **D** $(1-x)(3+x)$

For questions **32** to **39**, write an algebraic expression in factorised form for the shaded area in each of the figures.

32



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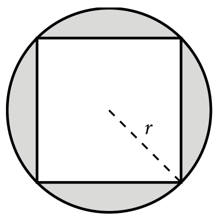

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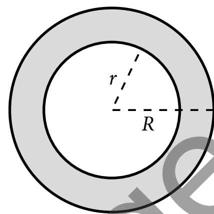

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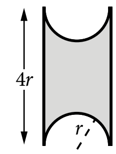

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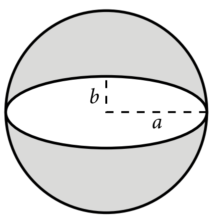

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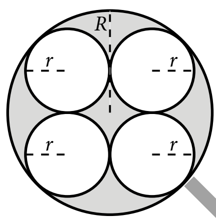

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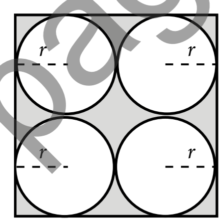

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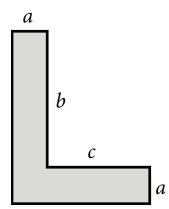


40 Indicate whether each answer is a correct or incorrect expression for the shaded area in question **39**.

(a) $(a+b)(a+c) - bc$ (b) $a^2 + ab + ac$ (c) $(a+b)(a+c)$ (d) $a(a+b+c)$

1.9 ALGEBRAIC FRACTIONS

To **simplify** algebraic fractions:

1 Factorise the numerator and denominator.

2 Cancel any common factors.

Example 11

Simplify:

(a) $\frac{9x+6}{3x+2}$ (b) $\frac{15a^2 - 5ab}{10ab}$ (c) $\frac{9x^2 - y^2}{6xy - 2y^2}$

Solution

(a) $\frac{9x+6}{3x+2} = \frac{3\cancel{(3x+2)}}{\cancel{3x+2}}$

$= 3$

(b) $\frac{15a^2 - 5ab}{10ab} = \frac{5a(3a-b)}{10ab}$

$= \frac{3a-b}{2b}$

(c) $\frac{9x^2 - y^2}{6xy - 2y^2} = \frac{\cancel{(3x-y)}(3x+y)}{2y\cancel{(3x-y)}}$

$= \frac{3x+y}{2y}$

Example 12

Simplify: $\dfrac{x^2-5x+6}{x^2-9}\times\dfrac{x^2+3x}{x^2-x-2}$

Solution

$$\frac{x^2-5x+6}{x^2-9}\times\frac{x^2+3x}{x^2-x-2}=\frac{\cancel{(x-2)}\,\cancel{(x-3)}}{\cancel{(x-3)}\,\cancel{(x+3)}}\times\frac{x\,\cancel{(x+3)}}{(x+1)\cancel{(x-2)}}$$

$$=\frac{x}{x+1}$$

EXERCISE 1.9 ALGEBRAIC FRACTIONS

Simplify:

1 $\dfrac{8a-4b}{4}$ **2** $\dfrac{15x+10y}{15}$ **3** $\dfrac{14x-7y}{2x-y}$ **4** $\dfrac{8x^2-4xy}{8xy}$

5 $\dfrac{12ab-6b^2}{9ab}$ **6** $\dfrac{8x+2}{4x+1}$ **7** $\dfrac{3a-5b}{3a^2-5ab}$ **8** $\dfrac{m+m^2}{m}$

9 $\dfrac{mn-n^2}{n}$ **10** $\dfrac{p^2q-pq^2}{pq}$ **11** $\dfrac{x^2+xy}{2x}$ **12** $\dfrac{2rs-12r}{r^2+rs}$

13 $\dfrac{x^2-y^2}{(x+y)^2}$ **14** $\dfrac{k^2+k}{k+1}$ **15** $\dfrac{x^2-9}{x^2+3x}$ **16** $\dfrac{15a^2-5ab}{3ab-b^2}$

17 $\dfrac{4x^2-4xy}{x^2-y^2}$ **18** $\dfrac{x^2-7x+6}{x^2-36}$ **19** $\dfrac{a^2+ab}{ab+b^2}$ **20** $\dfrac{a^2-b^2}{a^2+ab}$

21 $\dfrac{x^2-1}{x^2-5x+4}$ **22** $\dfrac{x^2-6x+8}{x^2-x-2}$ **23** $\dfrac{x^2+3x+2}{x^2-4}$ **24** $\dfrac{x^2-5x+6}{x^2+x-12}$

25 $\dfrac{x^2+4x+4}{x^2-3x-10}$ **26** $\dfrac{4x^3y-16xy}{x^2+2x-8}$ **27** $\dfrac{12a+9}{15}\times\dfrac{5}{4a+3}$ **28** $\dfrac{3x^2-xy}{xy}\times\dfrac{x^2y}{3xy-y^2}$

29 $\dfrac{m^2+m-2}{m^2-m}$ simplifies to:

A $\dfrac{(m-2)(m+1)}{m(m-1)}$ B 2 C $\dfrac{m+2}{m}$ D $\dfrac{m-2}{m}$

30 $\dfrac{2a^2-3ab}{ab-b^2}\times\dfrac{2a^2-2ab}{4a-6b}$ **31** $\dfrac{15x^2-5xy}{10xy}\div\dfrac{3x-y}{2y}$ **32** $\dfrac{12x^2-4x}{3x^2-x}\div\dfrac{10x^2y}{5x^2y^2}$

33 $\dfrac{x^2-2x-3}{x^2-4x-5}\times\dfrac{x^2-25}{(x-3)(x+5)}$ **34** $\dfrac{(a+2b)(a-b)}{a^2-4b^2}\times\dfrac{a^2-3ab+2b^2}{ab-b^2}$

35 Simplify $\dfrac{m^2-9}{m^2-m-12}\div\dfrac{m^2-3m}{m^2-9m+20}$. Indicate whether each answer is correct or incorrect.

(a) $\dfrac{m(m-3)^2}{(m-4)(m-5)}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{m+5}{m}$ (d) $\dfrac{m-5}{m}$

Simplify:

36 $\dfrac{x^3+y^3}{x^2-y^2}$ **37** $\dfrac{8x^2+4x+2}{8x^3-1}$ **38** $\dfrac{2x+2y}{x^3-y^3}\times\dfrac{x^2-2xy+y^2}{x^2-y^2}$

39 $\dfrac{(x+h)^3-x^3}{h}$ **40** $\dfrac{x^3-(x-y)^3}{x^2-(x-y)^2}$

1.10 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

When adding and subtracting fractions, first rewrite each fraction with the same (common) denominator, then add or subtract the numerators.

Example 13

Express as a single fraction:

(a) $\frac{3}{5}-\frac{2}{7}$ (b) $\frac{3x}{5}-\frac{2x+1}{7}$ (c) $\frac{3}{5x}+\frac{2}{7x}$ (d) $\frac{2x-y}{3}-\frac{x-y}{6}$

Solution

(a) $\frac{3}{5}-\frac{2}{7}$

$=\frac{21}{35}-\frac{10}{35}$

$=\frac{11}{35}$

(b) $\frac{3x}{5}-\frac{2x+1}{7}$

$=\frac{21x}{35}-\frac{5(2x+1)}{35}$

$=\frac{21x-10x-5}{35}$

$=\frac{11x-5}{35}$

(c) $\frac{3}{5x}+\frac{2}{7x}$

$=\frac{21}{35x}+\frac{10}{35x}$

$=\frac{31}{35x}$

(d) $\frac{2x-y}{3}-\frac{x-y}{6}$

$=\frac{2(2x-y)}{6}-\frac{(x-y)}{6}$

$=\frac{4x-2y-x+y}{6}$

$=\frac{3x-y}{6}$

Harder algebraic fractions

More-complex algebraic fractions require you to factorise the denominator before you find the common denominator. Write each fraction with the common denominator before you add or subtract the numerators.

Example 14

Express as a single fraction:

(a) $\frac{3}{x^2-4}+\frac{1}{x-2}$ (b) $\frac{1}{x-y}-\frac{1}{x+y}$ (c) $\frac{3}{x^2+2x}-\frac{2}{x^2-4}$ (d) $\frac{1}{x^2-5x+6}-\frac{1}{x^2+2x-8}$

Solution

(a) $\frac{3}{x^2-4}+\frac{1}{x-2}$

$=\frac{3}{(x-2)(x+2)}+\frac{1}{x-2}$

$=\frac{3}{(x-2)(x+2)}+\frac{x+2}{(x-2)(x+2)}$

$=\frac{x+5}{(x-2)(x+2)}$

(b) $\frac{1}{x-y}-\frac{1}{x+y}$

$=\frac{x+y}{(x-y)(x+y)}-\frac{x-y}{(x-y)(x+y)}$

$=\frac{x+y-(x-y)}{(x-y)(x+y)}$

$=\frac{2y}{(x-y)(x+y)}$

(c) $\frac{3}{x^2+2x}-\frac{2}{x^2-4}$

$=\frac{3}{x(x+2)}-\frac{2}{(x-2)(x+2)}$

$=\frac{3(x-2)}{x(x+2)(x-2)}-\frac{2x}{x(x-2)(x+2)}$

$=\frac{3x-6-2x}{x(x+2)(x-2)}$

$=\frac{x-6}{x(x+2)(x-2)}$

(d) $\frac{1}{x^2-5x+6}-\frac{1}{x^2+2x-8}$

$=\frac{1}{(x-2)(x-3)}-\frac{1}{(x-2)(x+4)}$

$=\frac{x+4}{(x-2)(x-3)(x+4)}-\frac{x-3}{(x-2)(x-3)(x+4)}$

$=\frac{x+4-(x-3)}{(x-2)(x-3)(x+4)}$

$=\frac{x+4-x+3}{(x-2)(x-3)(x+4)}$

$=\frac{7}{(x-2)(x-3)(x+4)}$

EXERCISE 1.10 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

Simplify:

1 $\frac{x}{5}-\frac{x}{6}$ **2** $\frac{3x}{8}+\frac{x}{2}$ **3** $\frac{a}{3}+\frac{4a}{5}-\frac{a}{6}$ **4** $\frac{y}{2}+\frac{2y}{3}-\frac{y}{4}$

5 $\frac{a+2}{5}-\frac{a-1}{3}$ **6** $\frac{2x-y}{3}-\frac{x-3y}{6}$ **7** $\frac{3x+2}{6}-\frac{x+1}{4}$ **8** $\frac{3m-2n}{5}+\frac{m+n}{10}$

9 $\frac{x}{2}+\frac{y}{4}-\frac{x+y}{3}$ **10** $\frac{a-2b}{6}-\frac{2a+b}{9}$ **11** $\frac{3(a+b)}{4}-\frac{a-b}{6}$ **12** $\frac{1}{x}-\frac{2}{3x}$

13 $\frac{3}{a}+\frac{1}{a^2}$ **14** $\frac{1}{ab}-\frac{2}{b}$ **15** $\frac{m}{n}-\frac{n}{m}$ **16** $\frac{4}{xy}+\frac{3}{yz}$

17 $\frac{5}{a^2b}-\frac{2}{ab^2}$ **18** $\frac{a+1}{6a}+\frac{a-4}{2a}$ **19** $\frac{1}{x+1}+\frac{2}{3}$ **20** $\frac{1}{x}+\frac{2}{x}-\frac{1}{x^2}$

21 Simplify $\frac{1}{ab}+\frac{a}{bc}$. Which of the following is correct?

A $\frac{1+a}{abc}$ **B** $\frac{c+a^2}{abc}$ **C** $\frac{1+a}{ab^2c}$ **D** $\frac{c+a^2}{ab^2c}$

For questions **22** to **29**, write the lowest common multiple (LCM).

22 $(x-3)$ and $(x+3)$ **23** x and $(x-2)$ **24** $(2x-4)$ and $(3x-6)$

25 (x^2-4x) and $(x-4)$ **26** (x^2-4x) and (x^2-16) **27** $(x+2)$ and (x^2+4x+4)

28 $(x-y)$, $(x+y)$, and (x^2-y^2) **29** (x^2-y^2), (x^2+xy), $(xy-y^2)$

30 The lowest common multiple of x, $x+3$ and x^2-9 is:

A $x(x-3)(x+3)$ **B** $x(x-3)(x^2-9)$ **C** $x(x+3)(x^2-9)$ **D** $x(x-3)^2$

Express each of the following as a single fraction.

31 $\frac{1}{a-b}+\frac{1}{a+b}$ **32** $\frac{3}{x-y}-\frac{2}{x+y}$ **33** $\frac{x}{x-y}+\frac{y}{x-y}$

34 $\frac{x}{x-y}+\frac{y}{x+y}$ **35** $\frac{3a-b}{a^2-b^2}+\frac{1}{a-b}$ **36** $\frac{1}{x^2-4}-\frac{1}{x+2}$

37 $\frac{x}{x^2-y^2}-\frac{y}{x^2-y^2}$ **38** $\frac{3}{(x-2)^2}+\frac{2}{x-2}$ **39** $\frac{1}{x^2-4x+3}-\frac{1}{x^2-1}$

40 $\frac{3}{x-2}+\frac{1}{x+3}$ **41** $\frac{1}{x+y}-\frac{1}{x-y}$ **42** $\frac{5}{2a+6}+\frac{a}{a^2-9}$

43 $\frac{1}{x-5}-\frac{1}{x+5}+\frac{x+10}{x^2-25}$ **44** $\frac{6}{3x-2}-\frac{8}{4x+1}$ **45** $\frac{7a}{3a-4}-\frac{5a}{2a-3}$

46 $\frac{y}{x^2-xy}+\frac{1}{x}$ **47** $\frac{1}{x+4}+\frac{x}{x^2-16}$ **48** $\frac{1}{x+2}+\frac{1}{x-2}+\frac{4}{x^2-4}$

49 $\frac{1}{a+3}+\frac{a+4}{a^2+5a+6}$ **50** $\frac{3}{x^2-4}-\frac{2}{x^2-3x+2}$ **51** $\frac{x+1}{x-1}-\frac{x-1}{x+1}$

52 $\frac{5}{x-2}+\frac{3}{x^2-4}$ **53** $\frac{3x}{x^2-16}-\frac{2}{x+4}$ **54** $\frac{5}{3x}-\frac{2}{x^2-5x}$

55 Simplify $\frac{5}{x+1}-\frac{3}{x-2}$. Indicate whether each answer is correct or incorrect.

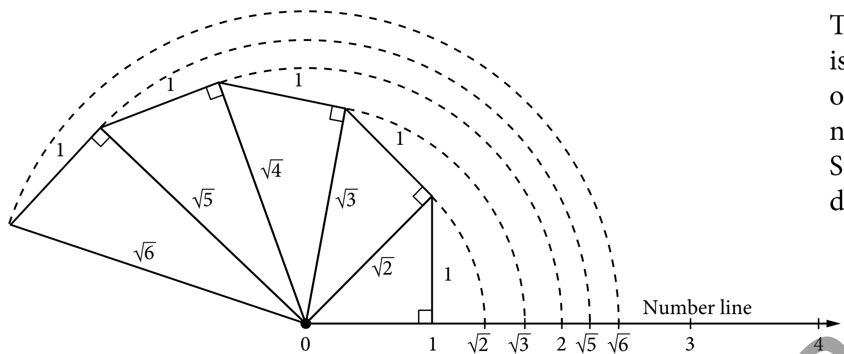
(a) $\frac{2x-7}{(x+1)(x-2)}$ **(b)** $\frac{2x-13}{(x+1)(x-2)}$ **(c)** $\frac{2}{(x+1)(x-2)}$ **(d)** $\frac{2x-1}{(x+1)(x-2)}$

1.11 REAL NUMBERS AND SURDS

Rational numbers may be expressed in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Rational numbers include fractions, terminating decimals and recurring decimals. Integers are rational numbers with a denominator of one, for example $5 = \frac{5}{1}$.

Irrational numbers are numbers that cannot be represented as a fraction, such as $\sqrt{2}$, $\sqrt[3]{15}$, $2\sqrt{7}$ or π. They are not rational. Irrational numbers that are roots, such as $\sqrt{2}$, $\sqrt[3]{15}$ and $2\sqrt{7}$, are called surds. Some other irrational numbers, such as π, are called transcendental numbers.

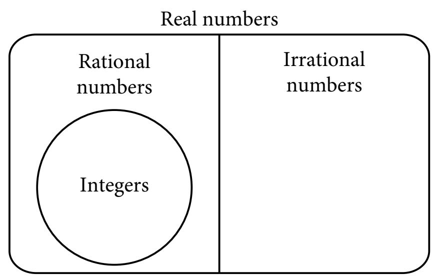
The following diagram shows how to construct exact lengths for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$ and $\sqrt{6}$, starting with a right-angled triangle with side lengths of one unit. Each triangle's hypotenuse is an exact value, written as a surd.



The diagram shows that $\sqrt{4} = 2$, which is a rational number. The square root of a perfect square is always a rational number. Similarly, $\sqrt[3]{8} = 2$ is rational. Surds that simplify to a rational number do not represent irrational numbers.

The sets of all rational and irrational numbers together form a larger set called the set of **real numbers**.

The real numbers can be represented by all the points on the real number line.



Operations with surds—basic rules

Surds have their own set of simplifying rules for the four operations of arithmetic. These rules are used to simplify expressions involving surds.

If a and b are positive numbers, then:

(a) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ **(b)** $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ **(c)** $\sqrt{a^2} = a$

(d) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ **(e)** $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ **(f)** $\sqrt{a} \times \sqrt{a} = a$

(g) $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{\sqrt{ab}}{b}$ to write the expression with a rational denominator.

Surds can also be simplified sometimes by identifying a factor in the surd that is a perfect square and then taking that factor's square root. (For example, see part **(a)** in Example **15**.) You might also multiply two surds and find that the answer has a perfect square factor.

You should know the perfect squares so that you can look for them as factors in surds. When a surd contains a perfect square factor, you can use rules **(a)**, **(c)** and **(g)** above to simplify the expression.

Example 15

Simplify each expression.

(a) $\sqrt{12}$ (b) $\sqrt{80}$ (c) $3\sqrt{28}$ (d) $\frac{\sqrt{175}}{5}$

(e) $\sqrt{6}\times\sqrt{10}$ (f) $\frac{\sqrt{80}}{\sqrt{5}}$ (g) $\sqrt{3}\times\sqrt{12}$ (h) $4\sqrt{2}\times\sqrt{24}$

Solution

(a) Look for a factor of 12 that is a perfect square.

$12 = 4 \times 3$

$\therefore \sqrt{12} = \sqrt{4\times3} = \sqrt{4}\times\sqrt{3} = 2\sqrt{3}$

(b) Look for a factor of 80 that is a perfect square.

$80 = 16 \times 5$

$\therefore \sqrt{80} = \sqrt{16\times5} = 4\sqrt{5}$

(c) $3\sqrt{28} = 3\sqrt{4}\times\sqrt{7} = 6\sqrt{7}$

(d) $\frac{\sqrt{175}}{5} = \frac{\sqrt{25\times7}}{5} = \frac{5\sqrt{7}}{5} = \sqrt{7}$

(e) $\sqrt{6}\times\sqrt{10} = \sqrt{60} = \sqrt{4\times15} = 2\sqrt{15}$

(f) $\frac{\sqrt{80}}{\sqrt{5}} = \sqrt{\frac{80}{5}} = \sqrt{16} = 4$

(g) $\sqrt{3}\times\sqrt{12} = \sqrt{3}\times2\sqrt{3} = 2\times3 = 6$

(h) $4\sqrt{2}\times\sqrt{24} = 4\sqrt{2}\times2\sqrt{6} = 8\sqrt{12} = 16\sqrt{3}$

Example 16

Simplify each expression, writing your answer with a rational denominator (or no denominator).

(a) $\frac{6}{\sqrt{3}}$ (b) $\frac{\sqrt{60}}{3\sqrt{6}}$ (c) $\frac{3\sqrt{5}}{\sqrt{15}}$

Solution

(a) $\frac{6}{\sqrt{3}} = \frac{6}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

(b) $\frac{\sqrt{60}}{3\sqrt{6}} = \frac{\sqrt{6}\times\sqrt{10}}{3\sqrt{6}}$

$= \frac{\sqrt{10}}{3}$

(c) $\frac{3\sqrt{5}}{\sqrt{15}} = \frac{3\sqrt{5}}{\sqrt{3}\times\sqrt{5}}$

$= \frac{3}{\sqrt{3}}$

$= \frac{\sqrt{3}\times\sqrt{3}}{\sqrt{3}}$

$= \sqrt{3}$

EXERCISE 1.11 REAL NUMBERS AND SURDS

Simplify all the expressions in this exercise, writing each answer with a rational denominator (or no denominator).

1 $\sqrt{8}$ **2** $\sqrt{20}$ **3** $\sqrt{27}$ **4** $\sqrt{32}$

5 When simplified, $\sqrt{40}$ equals:

A $4\sqrt{5}$ **B** $\sqrt{10}$ **C** $2\sqrt{10}$ **D** 20

6 $\sqrt{45}$ **7** $\sqrt{72}$ **8** $\sqrt{84}$ **9** $\sqrt{98}$ **10** $\sqrt{108}$ **11** $\sqrt{125}$

12 $\sqrt{162}$ **13** $\sqrt{200}$ **14** $5\sqrt{128}$ **15** $4\sqrt{800}$ **16** $2\sqrt{150}$ **17** $3\sqrt{52}$

18 When simplified, $7\sqrt{245}$ equals:

A $7\sqrt{5}$ **B** $35\sqrt{7}$ **C** $35\sqrt{5}$ **D** $49\sqrt{5}$

19 $\frac{\sqrt{320}}{2}$ **20** $\frac{\sqrt{175}}{5}$ **21** $\frac{4\sqrt{243}}{3}$ **22** $\frac{\sqrt{90}}{9}$

23 When simplified, $\frac{\sqrt{50}}{\sqrt{5}}$ equals:

A $\sqrt{2}$ B $\sqrt{10}$ C $5\sqrt{2}$ D 10

24 Simplify $\frac{2\sqrt{7}}{\sqrt{35}}$. Indicate whether each answer below is correct or incorrect.

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{14}{\sqrt{5}}$ (c) $\frac{2\sqrt{5}}{5}$ (d) 10

25 $\sqrt{3}\times\sqrt{5}$ 26 $\sqrt{8}\times\sqrt{2}$ 27 $\sqrt{6}\times\sqrt{2}$ 28 $\sqrt{6}\times\sqrt{10}$ 29 $2\sqrt{3}\times 4\sqrt{2}$

30 $\sqrt{8}\times 2\sqrt{2}$ 31 $2\sqrt{5}\times 5\sqrt{2}$ 32 $4\sqrt{6}\times 2\sqrt{3}$ 33 $2\sqrt{5}\times 3\sqrt{7}$ 34 $4\sqrt{3}\times\sqrt{18}$

35 $2\sqrt{8}\times\sqrt{12}$ 36 $4\sqrt{5}\times\sqrt{20}$ 37 $\frac{\sqrt{3}}{\sqrt{2}}$ 38 $\frac{2}{\sqrt{3}}$ 39 $\frac{\sqrt{28}}{\sqrt{7}}$

40 $\frac{7}{\sqrt{7}}$ 41 $\frac{\sqrt{80}}{\sqrt{5}}$ 42 $\frac{3\sqrt{2}}{\sqrt{6}}$ 43 $\frac{7\sqrt{2}}{\sqrt{98}}$ 44 $\frac{2\sqrt{18}}{\sqrt{8}}$

1.12 ADDING AND SUBTRACTING SURDS

Surds of the same kind can be added and subtracted just like pronumerals, using the distributive law to collect like terms: $ab + ac = a(b + c)$

You can only add and subtract like algebraic terms, such as $2a + 3a - a = 4a$. Similarly, you can only add and subtract like surds, such as $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$. It may be necessary to first simplify the surd terms by removing perfect square factors.

Example 17

Simplify each expression by collecting like terms.

(a) $2\sqrt{3}+5\sqrt{3}$ (b) $4\sqrt{10}-\sqrt{10}$ (c) $3\sqrt{6}+4\sqrt{6}-\sqrt{5}$ (d) $3\sqrt{5}+4\sqrt{7}+2\sqrt{5}-6\sqrt{7}$

Solution

(a) $2\sqrt{3}+5\sqrt{3}=(2+5)\sqrt{3}=7\sqrt{3}$

(b) $4\sqrt{10}-\sqrt{10}=(4-1)\sqrt{10}=3\sqrt{10}$

(c) $3\sqrt{6}+4\sqrt{6}-\sqrt{5}=7\sqrt{6}-\sqrt{5}$

(d) $3\sqrt{5}+4\sqrt{7}+2\sqrt{5}-6\sqrt{7}=5\sqrt{5}-2\sqrt{7}$

In parts **(c)** and **(d)** the different surds cannot be combined into a single term.

Example 18

Simplify:

(a) $\sqrt{8}-\sqrt{18}+\sqrt{50}$ (b) $5\sqrt{3}+\sqrt{20}-2\sqrt{12}+\sqrt{45}$

Solution

Where possible, simplify each term before attempting to add or subtract the surds.

(a) $\sqrt{8}-\sqrt{18}+\sqrt{50}$
$=\sqrt{4\times 2}-\sqrt{9\times 2}+\sqrt{25\times 2}$
$=2\sqrt{2}-3\sqrt{2}+5\sqrt{2}$
$=4\sqrt{2}$

(b) $5\sqrt{3}+\sqrt{20}-2\sqrt{12}+\sqrt{45}$
$=5\sqrt{3}+2\sqrt{5}-4\sqrt{3}+3\sqrt{5}$
$=\sqrt{3}+5\sqrt{5}$

EXERCISE 1.12 ADDING AND SUBTRACTING SURDS

Simplify the expressions in this exercise.

1 $\sqrt{3}+2\sqrt{3}+4\sqrt{3}$

2 $5\sqrt{7}-2\sqrt{7}+4\sqrt{7}$

3 $3\sqrt{5}+5\sqrt{5}-2\sqrt{5}$

4 $4\sqrt{2}-\sqrt{3}+4\sqrt{3}-\sqrt{2}$

5 $\sqrt{5}+\sqrt{2}+3\sqrt{5}-6\sqrt{2}$

6 $\sqrt{8}-\sqrt{2}$

7 When $\sqrt{27}-\sqrt{18}+\sqrt{3}$ is simplified, the answer is:

A $4\sqrt{3}-3\sqrt{2}$ B $\sqrt{3}$ C $3+\sqrt{3}$ D $2\sqrt{3}$

8 $\sqrt{20}+\sqrt{5}$

9 $\sqrt{18}+\sqrt{32}-\sqrt{2}$

10 $\sqrt{27}+2\sqrt{48}-4\sqrt{3}$

11 $\sqrt{12}+\sqrt{3}+\sqrt{48}$

12 $2\sqrt{50}-3\sqrt{18}$

13 $\sqrt{7}+\sqrt{28}-\sqrt{63}$

14 Simplify $\sqrt{5}+\sqrt{2}-\sqrt{45}+\sqrt{8}$. Some steps in this simplification are given below. Indicate whether each statement is a correct or incorrect step.

(a) $\sqrt{10}-\sqrt{45}$ (b) $-\sqrt{35}$ (c) $\sqrt{5}+\sqrt{2}-3\sqrt{5}+2\sqrt{2}$ (d) $3\sqrt{2}-2\sqrt{5}$

15 $6\sqrt{5}+4\sqrt{7}-2\sqrt{5}$

16 $5\sqrt{3}+\sqrt{27}-\sqrt{45}$

17 $\sqrt{20}+\sqrt{5}+\sqrt{18}$

18 $3\sqrt{15}+\sqrt{60}-\sqrt{40}$

19 $4\sqrt{7}-\sqrt{28}+\sqrt{63}$

20 $2\sqrt{50}-3\sqrt{18}+\sqrt{3}$

21 $\sqrt{3}+3\sqrt{75}-\sqrt{48}$

22 $\sqrt{150}-\sqrt{200}$

23 $\sqrt{6}+\sqrt{24}+\sqrt{54}$

24 $5\sqrt{7}+3\sqrt{5}-2\sqrt{28}$

25 $5\sqrt{45}-2\sqrt{32}$

26 $\sqrt{98}-2\sqrt{20}-\sqrt{12}$

27 $\sqrt{125}-5\sqrt{2}+\sqrt{50}$

28 $7\sqrt{3}-2\sqrt{2}+\sqrt{12}+\sqrt{8}$

29 $\sqrt{150}-\sqrt{96}-\sqrt{24}$

30 $6\sqrt{2}+\sqrt{12}-2\sqrt{3}-3\sqrt{8}$

1.13 THE DISTRIBUTIVE LAW

The distributive law can be used with surds to expand expressions with a binomial factor: $a(b + c) = ab + ac$

Example 19

Expand and simplify:

(a) $\sqrt{6}\left(\sqrt{2}+2\sqrt{3}\right)$ (b) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{6}\right)$ (c) $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$ (d) $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$

Solution

(a) $\sqrt{6}\left(\sqrt{2}+2\sqrt{3}\right)$

$=\sqrt{6}\times\sqrt{2}+\sqrt{6}\times 2\sqrt{3}$

$=\sqrt{12}+2\sqrt{18}$

$=2\sqrt{3}+6\sqrt{2}$

(b) $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{6}\right)$

$=\sqrt{3}\left(\sqrt{5}+\sqrt{6}\right)+\sqrt{2}\left(\sqrt{5}+\sqrt{6}\right)$

$=\sqrt{15}+3\sqrt{2}+\sqrt{10}+2\sqrt{3}$

(c) $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$

$=\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2$

$=5-3$

$=2$

This is similar to $(a-b)(a+b)=a^2-b^2$.

(d) $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$

$=\sqrt{5}\left(\sqrt{5}+\sqrt{3}\right)+\sqrt{3}\left(\sqrt{5}+\sqrt{3}\right)$

$=5+2\sqrt{15}+3$

$=8+2\sqrt{15}$

This is similar to $(a+b)^2=a^2+2ab+b^2$.

EXERCISE 1.13 THE DISTRIBUTIVE LAW

Expand and simplify the expressions in this exercise.

1 $\sqrt{5}\left(\sqrt{2}+\sqrt{3}\right)$ **2** $\sqrt{5}\left(\sqrt{5}+\sqrt{2}\right)$ **3** $\sqrt{2}\left(\sqrt{2}+\sqrt{8}\right)$

4 $\sqrt{3}\left(\sqrt{2}-\sqrt{6}\right)$ **5** $\sqrt{6}\left(\sqrt{3}-2\right)$ **6** $7\left(2\sqrt{5}-1\right)$

7 When simplified, $\sqrt{2}\left(\sqrt{32}-\sqrt{8}\right)$ equals:

A $8\sqrt{3}$ B 4 C $2\sqrt{2}$ D $8-2\sqrt{2}$

8 $3\sqrt{2}\left(2\sqrt{6}-\sqrt{5}\right)$ **9** $\sqrt{a}\left(\sqrt{a}+\sqrt{b}\right)$ **10** $\sqrt{x}\left(\sqrt{x}-\sqrt{y}\right)$

11 $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{2}\right)$ **12** $\left(\sqrt{2}+\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{2}\right)$ **13** $\left(\sqrt{3}-1\right)\left(\sqrt{2}+3\right)$

14 $\left(\sqrt{5}+2\right)\left(2\sqrt{5}+3\right)$ **15** $\left(2\sqrt{3}-5\right)\left(2\sqrt{3}+3\right)$ **16** $\left(\sqrt{3}-\sqrt{2}\left(2\sqrt{3}-\sqrt{2}\right)\right)$

17 $\left(2\sqrt{5}-\sqrt{2}\right)\left(2\sqrt{5}+3\right)$ **18** $\left(2\sqrt{2}-\sqrt{6}\right)\left(2\sqrt{3}-1\right)$ **19** $\left(\sqrt{3}+1\right)^2$

20 $\left(\sqrt{5}-\sqrt{2}\right)^2$ **21** $\left(2\sqrt{6}+\sqrt{3}\right)^2$ **22** $\left(2\sqrt{2}-1\right)\left(2\sqrt{2}+1\right)$

23 $\left(2\sqrt{6}-\sqrt{3}\right)\left(2\sqrt{6}+\sqrt{3}\right)$ **24** $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$ **25** $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$

26 When simplified, $\left(3\sqrt{7}-2\right)^2$ equals:

A 59 B $67+12\sqrt{7}$ C $23-12\sqrt{7}$ D $67-12\sqrt{7}$

27 $\left(\sqrt{5}-\sqrt{3}\right)^2$ **28** $\left(\sqrt{11}+\sqrt{7}\right)^2$ **29** $\left(2\sqrt{3}-1\right)\left(\sqrt{3}+2\right)$

30 $\left(\sqrt{11}-\sqrt{10}\right)\left(\sqrt{11}+\sqrt{10}\right)$ **31** $\left(\sqrt{6}-\sqrt{5}\right)\left(\sqrt{6}+\sqrt{5}\right)$ **32** $\left(2\sqrt{2}+\sqrt{3}\right)^2$

33 $\left(3\sqrt{5}-2\sqrt{2}\right)\left(3\sqrt{5}+2\sqrt{2}\right)$ **34** $\left(\sqrt{5}+2\sqrt{2}\right)\left(\sqrt{6}-1\right)$ **35** $\left(2\sqrt{6}-\sqrt{3}\right)\left(\sqrt{6}+3\sqrt{3}\right)$

36 Expand and simplify $\left(4\sqrt{3}+1\right)\left(2\sqrt{3}-3\right)$. Some steps in this simplification are given below. Indicate whether each statement is a correct or incorrect step.

(a) $72-12\sqrt{3}+2\sqrt{3}-3$ (b) $24-12\sqrt{3}+2\sqrt{3}-3$ (c) $21-10\sqrt{3}$ (d) $27-10\sqrt{3}$

37 $\left(5\sqrt{2}-4\right)\left(5\sqrt{2}+4\right)$ **38** $\left(2\sqrt{7}+3\sqrt{6}\right)^2$ **39** $\left(2\sqrt{15}+\sqrt{5}\right)\left(\sqrt{15}-3\sqrt{5}\right)$

40 $\left(2\sqrt{2}+3\sqrt{3}\right)^2$ **41** $\left(2\sqrt{3}-1\right)\left(2\sqrt{3}+1\right)$ **42** $\left(2\sqrt{3}-1\right)^2$

1.14 RATIONALISING DENOMINATORS

The expressions $\sqrt{a}\times\sqrt{a}$ and $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$ both have rational answers (that is, answers that do not involve surds): $\sqrt{a}\times\sqrt{a}=a$ and $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2=a-b$. For example:

- $\sqrt{3}\times\sqrt{3}=3$
- $\left(\sqrt{7}-\sqrt{2}\right)\left(\sqrt{7}+\sqrt{2}\right)=7-2=5$

The expansion $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)=a-b$ is known as the 'difference of two squares'. You have seen this previously as $(x-y)(x+y)=x^2-y^2$. By letting $x=\sqrt{a}$ and $y=\sqrt{b}$, we have obtained a process to convert any binomial surd into a rational number.

When you have a surd expression in the denominator of a fraction, it is normal to make the denominator into a rational number. This process is called **rationalising the denominator**.

Remember: to change a fraction without changing its value, multiply the numerator and the denominator by the same amount.

The expressions $\sqrt{a}-\sqrt{b}$ and $\sqrt{a}+\sqrt{b}$ are called **conjugate** surds, with each expression being the conjugate of the other. To convert any surd into a rational number, multiply the surd by its conjugate.

Example 20

Express with a rational denominator:

(a) $\frac{1}{\sqrt{2}}$ **(b)** $\frac{\sqrt{7}}{\sqrt{3}}$ **(c)** $\frac{1}{\sqrt{3}+\sqrt{2}}$ **(d)** $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}$

Solution

(a) $\frac{1}{\sqrt{2}}$: multiply by $\frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

(b) $\frac{\sqrt{7}}{\sqrt{3}}$: multiply by $\frac{\sqrt{3}}{\sqrt{3}}$

$$= \frac{\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{21}}{3}$$

In parts **(c)** and **(d)** the denominator is a binomial surd, so you have to multiply both numerator and denominator by the conjugate of the denominator.

(c) $$\frac{1}{\sqrt{3}+\sqrt{2}} = \frac{1}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{3-2}$$

$$= \sqrt{3}-\sqrt{2}$$

(d) $$\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}} \times \frac{2\sqrt{7}+\sqrt{3}}{2\sqrt{7}+\sqrt{3}}$$

$$= \frac{2\sqrt{21}+3}{4\times 7-3}$$

$$= \frac{2\sqrt{21}+3}{25}$$

Example 21

Express $\frac{\sqrt{2}}{2\sqrt{2}+1} + \frac{2}{\sqrt{3}+1}$ as a single fraction with a rational denominator.

Solution

You can add the two fractions by finding a common denominator and then rationalising the result, or by first rationalising each denominator and then adding the resulting fractions. The latter is usually the easier method, unless the denominators happen to be conjugates.

$$\frac{\sqrt{2}}{2\sqrt{2}+1}+\frac{2}{\sqrt{3}+1}=\frac{\sqrt{2}}{2\sqrt{2}+1}\times\frac{2\sqrt{2}-1}{2\sqrt{2}-1}+\frac{2}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$
$$=\frac{4-\sqrt{2}}{4\times 2-1}+\frac{2\left(\sqrt{3}-1\right)}{3-1}$$
$$=\frac{4-\sqrt{2}}{7}+\frac{2\left(\sqrt{3}-1\right)}{2}$$
$$=\frac{4-\sqrt{2}}{7}+\frac{\sqrt{3}-1}{1}$$
$$=\frac{4-\sqrt{2}+7\left(\sqrt{3}-1\right)}{7}$$
$$=\frac{7\sqrt{3}-\sqrt{2}-3}{7}$$

EXERCISE 1.14 RATIONALISING DENOMINATORS

For questions **1** to **27**, express each fraction with a rational denominator.

1 $\frac{2}{\sqrt{3}}$ **2** $\frac{\sqrt{5}}{\sqrt{3}}$ **3** $\frac{3\sqrt{5}}{\sqrt{15}}$ **4** $\frac{1}{\sqrt{3}-\sqrt{2}}$

5 $\frac{1}{2\sqrt{7}+\sqrt{6}}$ **6** $\frac{1}{\sqrt{5}+2}$ **7** $\frac{1}{2\sqrt{5}-3\sqrt{2}}$ **8** $\frac{3\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

9 $\frac{\sqrt{6}}{2\sqrt{5}-3\sqrt{2}}$ **10** $\frac{2\sqrt{5}}{3\sqrt{11}-2\sqrt{8}}$ **11** $\frac{\sqrt{3}+2}{\sqrt{3}-2}$ **12** $\frac{4\sqrt{2}+3\sqrt{5}}{2\sqrt{5}-\sqrt{2}}$

13 $\frac{\sqrt{7}-2\sqrt{5}}{3\sqrt{5}-2\sqrt{2}}$ **14** $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$ **15** $\frac{5\sqrt{3}+3\sqrt{5}}{5\sqrt{5}-3\sqrt{3}}$ **16** $\frac{2\sqrt{3}}{3\sqrt{3}-2}$

17 $\frac{3\sqrt{6}}{2\sqrt{2}+\sqrt{3}}$ **18** $\frac{5\sqrt{11}+3}{3\sqrt{11}-2}$ **19** $\frac{\sqrt{2}}{2\sqrt{2}-2\sqrt{3}}$ **20** $\frac{3\sqrt{3}}{2\sqrt{3}+\sqrt{2}}$

21 $\frac{\sqrt{3}}{\sqrt{24}-\sqrt{48}}$ **22** $\frac{\sqrt{2}-1}{\sqrt{2}+1}$ **23** $\frac{2\sqrt{5}-\sqrt{2}}{2\sqrt{5}+\sqrt{2}}$ **24** $\frac{\sqrt{6}+2\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$

25 $\frac{2\sqrt{7}-3\sqrt{2}}{2\sqrt{7}-\sqrt{2}}$ **26** $\frac{\sqrt{5}+\sqrt{3}}{2\sqrt{10}-\sqrt{6}}$ **27** $\frac{2\sqrt{2}-5\sqrt{3}}{3\sqrt{6}-\sqrt{15}}$

28 If $x=\frac{3-2\sqrt{2}}{3+2\sqrt{2}}$ then the value of $x+\frac{1}{x}$ is:

A 34 **B** $24\sqrt{2}$ **C** $34-24\sqrt{2}$ **D** 2

29 If $x=\sqrt{3}+1$, find the value of $x^2-\frac{1}{x^2}$.

30 If $x=3\sqrt{2}+1$, find the value of $\frac{x^2-2x}{x-1}$.

31 If $x=\sqrt{5}-2$, find the value of $\frac{x^2+2x}{x+3}$.

32 Show that $x=2\sqrt{2}-3$ is one solution of the equation $x^2+6x+1=0$.

33 Which values of x satisfy $x^2+4x+2=0$? Indicate whether each answer is correct or incorrect.

(a) $x=2-\sqrt{2}$ (b) $x=\sqrt{2}-2$ (c) $x=\sqrt{2}+2$ (d) $x=-2-\sqrt{2}$

34 Show that $x = \sqrt{5} - 1$ is one solution of the equation $x^3 + 3x^2 - 2x - 4 = 0$.

35 Show that $x = \dfrac{1}{\sqrt{3} - 1}$ is one solution of the equation $2x^2 - 2x - 1 = 0$.

For questions **36** to **44**, express as a single fraction with a rational denominator:

36 $\dfrac{1}{2\sqrt{3} - 1} + \dfrac{3}{\sqrt{3} + 1}$

37 $\dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} - \dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

38 $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} \times \dfrac{2\sqrt{2} - \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$

39 $\dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} \times \dfrac{2\sqrt{5} - 3\sqrt{2}}{\sqrt{5} + 2\sqrt{2}}$

40 $\dfrac{2\sqrt{5} + 1}{2\sqrt{5} - 1} - \dfrac{\sqrt{5} - 1}{2\sqrt{5} - 3}$

41 $\dfrac{2\sqrt{3}}{\sqrt{6} - \sqrt{3}} - \dfrac{\sqrt{3}}{2\sqrt{6} + 3\sqrt{3}}$

42 $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 2} - \dfrac{\sqrt{5} - \sqrt{3}}{2\sqrt{5} + \sqrt{3}}$

43 $\dfrac{2\sqrt{5}}{\sqrt{10} - \sqrt{15}} - \dfrac{3\sqrt{7}}{\sqrt{35} - \sqrt{14}}$

44 $\dfrac{1}{x - 1} + \dfrac{1}{x + 1} - \dfrac{2}{x^2 - 1}$, where $x = 2\sqrt{3} + 1$.

CHAPTER REVIEW 1

1 Expand and simplify:

(a) $(2x - 5)^2$ **(b)** $(x + 3)(x - 7)$ **(c)** $(2y + 1)(3y + 4)$
(d) $(5x - 4)(5x + 4)$ **(e)** $(2x - y)(x^2 - xy + y^2)$

2 Factorise:

(a) $81 - 4a^2$ **(b)** $10xy - 5y$ **(c)** $5x^2y - 10xy^2 - 5xy$ **(d)** $a^2 - 18a + 56$
(e) $16x^2 - 1$ **(f)** $3x^2 + 4x - 7$ **(g)** $a^2 - b^2 + 2a - 2b$ **(h)** $3x + 7x^2 - 6x^3$
(i) $8a^3 - 27$ **(j)** $8 - (x + h)^3$ **(k)** $8y^2 - 6y - 9$ **(l)** $x^4 - 8x$

3 If $a = -3$ and $b = 4$, evaluate $\sqrt{a^2 + b^2}$.

4 Simplify:

(a) $\dfrac{3x^3}{4a^2} \times \dfrac{ay - a}{xy^2} \div \dfrac{3y - 3}{4ay^2}$ **(b)** $\dfrac{5x}{3} - \dfrac{2x + 3}{4} + \dfrac{x}{6}$ **(c)** $\dfrac{12m^2 - 4n}{3m^2 - n} \div \dfrac{10m^2n}{5m^2n^2}$

(d) $\dfrac{3x + 4}{x^2} - \dfrac{5}{x}$ **(e)** $\dfrac{3}{x + 1} + \dfrac{1}{x^2 + 2x + 1}$ **(f)** $\dfrac{2x^2 - 3xy}{xy - y^2} \div \dfrac{4x - 6y}{2x^2 - 2xy}$

(g) $\dfrac{a^3 + b^3}{a^2 - b^2}$ **(h)** $\dfrac{2}{m^2 - 4} - \dfrac{1}{m^2 - 3m + 2}$ **(i)** $\dfrac{1}{x^2 - 9x + 20} + \dfrac{1}{x^2 - 11x + 30}$

5 Expand and simplify:

(a) $\left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$ **(b)** $\left(a^{\frac{1}{2}} - b^{-\frac{1}{2}}\right)^2$

6 Find the exact value of $\dfrac{a^4b}{c^4}$ where $a = \left(\dfrac{2}{3}\right)^2$, $b = \left(\dfrac{8}{3}\right)^7$, $c = \left(\dfrac{4}{3}\right)^4$.

7 If $E = \dfrac{m}{2}\left(v^2 - u^2\right)$, find:

(a) E when $m = 13$, $v = 17$, $u = 15$ **(b)** v when $E = 98$, $m = 4$, $u = 24$, if $v > 0$

8 Simplify:

(a) $\sqrt{12} + 4\sqrt{3} + \sqrt{27} - 3\sqrt{54}$ **(b)** $\dfrac{2}{\sqrt{5}} + \sqrt{20} + \dfrac{8}{\sqrt{80}}$

9 **(a)** $\dfrac{\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$ **(b)** $\dfrac{3^{\frac{5}{4}} \times 15^{\frac{3}{4}}}{5^{-\frac{3}{4}} \times 45^{\frac{1}{2}}}$ (A challenge!)

10 Expand and simplify $\left(3\sqrt{5}-2\sqrt{3}\right)^2$.

11 The correct expansion of $\left(\sqrt{5}+2\sqrt{2}\right)\left(\sqrt{6}-\sqrt{5}\right)$ is:

A $\sqrt{30}+2\sqrt{2}-5$ **B** $\sqrt{6}+2\sqrt{12}-2\sqrt{10}$ **C** $4\sqrt{3}-\sqrt{10}-5$ **D** $\sqrt{30}+4\sqrt{3}-2\sqrt{10}-5$

12 Express $\dfrac{\sqrt{2}-1}{2\sqrt{2}-1}$ with a rational denominator.

13 Find the exact value of x^4-2x^2+1 when $x=3\sqrt{2}$.

14 Show by substitution that $x=\dfrac{\sqrt{5}-1}{2}$ is a root of the equation $x^3+3x^2+x-2=0$.

15 Simplify:

(a) $1-\dfrac{1}{1-\left(-\frac{1}{4}\right)}$ **(b)** $\dfrac{8^2+6^2-4^2}{2\times 8\times 6}$ **(c)** $H\pi\left(\dfrac{2R}{3}\right)^2-\dfrac{H}{R}\pi\left(\dfrac{2R}{3}\right)^3$ **(d)** $\dfrac{2x(x-1)-\left(x^2+3\right)\times 1}{(x-1)^2}$

(e) $\dfrac{1}{36}\times\dfrac{\left(\frac{35}{36}\right)^2}{1-\left(\frac{35}{36}\right)^2}$ **(f)** $2+\dfrac{t^2}{t+1}-\left(1+\dfrac{1}{t+1}\right)$ **(g)** $2\left(\dfrac{8}{3}+2-\left(\dfrac{1}{3}+1\right)\right)$

(h) Simplify, factorising your answer as much as possible: $\left(\sqrt{(x+1)^2+(y-0)^2}\right)^2+\left(\sqrt{(x-3)^2+(y-0)^2}\right)^2=40$

16 Factorise fully:

(a) $2x^2-8$ **(b)** $6x^2-216$

17 Evaluate:

(a) $100\,000\times 1.006^{120}-780\left(\dfrac{1.006^{120}-1}{0.006}\right)$

(b) correct to 2 decimal places, $M=\dfrac{500\,000\times 1.005^{360}\times 0.005}{1\left(1.005^{360}-1\right)}$

(c) correct to 2 decimal places, $A_{20}=500\,000\times 1.005^{240}-2998\times\dfrac{1.005^{240}-1}{0.005}$

(d) $\$500\times\dfrac{1.003^{60}\left(\left(\frac{1.01}{1.003}\right)^{60}-1\right)}{\frac{1.01}{1.003}-1}$

18 Make y the subject of the formula in $\dfrac{H}{H-y}=\dfrac{R}{x}$.

19 If $\dfrac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{n+1}-\sqrt{n}$ then find the value of $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\dfrac{1}{\sqrt{99}+\sqrt{100}}$.