NSW
STAGE 6

NEW SENIOR
Mathematics
ADVANCED + EXTENSION 1 YEAR 12 | 4TH EDITION

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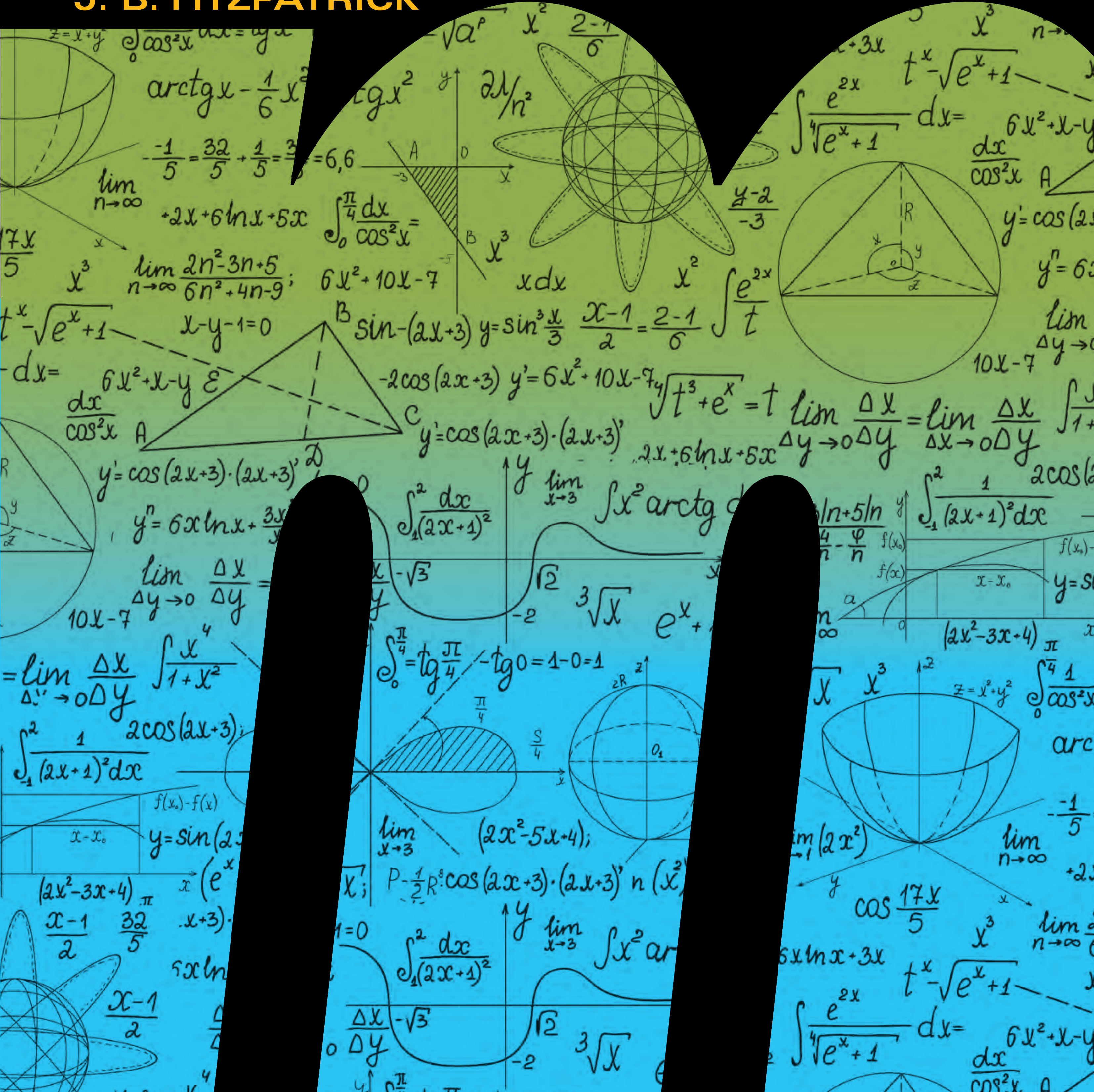


CHAPTER 14
Differential equations

Bernie Fitzpatrick, the original author of this series, once wrote: ‘Mathematics is about doing, not about reading’. This chapter brings together many skills learnt across the Mathematics Advanced and Extension 1 courses and asks you to apply them in practical situations.

Differential equations involve derivatives, so to be able to solve them you must be proficient at integration. These integrations often involve the exponential and logarithmic functions, so you must know the rules needed to manipulate these functions. You must be confident with a high level of algebraic manipulation, sometimes involving several levels of algebraic fractions. When integrating the trigonometric functions you need to be able to apply the appropriate Pythagorean identities, as well as the appropriate double angle formulae. If the integration required is beyond this course, then appropriate guidance is given in the question.

14.1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

The study of differential equations began in the late 1600s with Sir Isaac Newton’s investigation of the orbits of the planets about the Sun. Newton referred to these equations as ‘fluxional equations’. The term **differential equation** was suggested by Newton’s contemporary, Gottfried Leibniz, who did much of the early work on them.

A differential equation is an equation that relates some unknown differentiable function to one or more of its derivatives.

For example, the general form of the first-order differential equation $y'(t) = f\left(t, y'(t)\right)$ expresses the rate of change of a quantity $y(t)$ in terms of two variables: the time t and the value of the quantity $y(t)$ itself.

Differential equations are a powerful way to represent, understand and predict the behaviour of variable quantities, including systems that change with time.

Whereas the solution of an algebraic equation such as $x + 1 = 0$ is a number, the solution of a differential equation is a function. More specifically, the solution of a differential equation will be a differentiable function $y = g(x)$ if the differential equation is true when y and its derivatives are replaced with $g(x)$ and its derivatives.

A solution of the differential equation $\frac{dy}{dx} = f(x, y)$ is any differentiable function $y = g(x)$ with a derivative $\frac{dy}{dx} = g'(x)$ so that $g'(x) = f\left(x, g(x)\right)$ for all $x \in (a, b)$.

In other words, substituting the solution $y = g(x)$ into the differential equation will reduce $\frac{dy}{dx} = f(x, y)$ to an identity in the independent variable x, for all values of this variable in some open interval of the x-axis. Therefore:

To verify that a function $y = g(x)$ is a solution of a differential equation, you can substitute the function and its derivative(s) into both sides of the differential equation and check that both sides are identically equal.

Verification of a solution to a differential equation

Example 1

Verify by differentiation that $y = 3e^{-x}$ is a solution of the differential equation $\frac{dy}{dx} = -y$.

Solution

Calculate the LHS of the equation: $\text{LHS} = \frac{dy}{dx}$

$$= 3\left(-e^{-x}\right)$$

$$= -3e^{-x}$$

Calculate the RHS of the equation: $\text{RHS} = -y$

$$= -3e^{-x}$$

The LHS of the equation is identically equal to the RHS of the equation for all relevant values of the independent variable.

Graphing particular members of a general solution

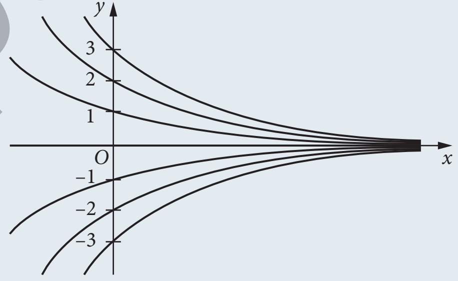
Example 2

Graph the following members of the one-parameter general solution $y = Ae^{-x}$ with $A \in \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution

The graph is drawn for $A = -3$, i.e., $y = -3e^{-x}$.

Next, separate graphs are drawn on the same axes for the other given values of A: $y = -2e^{-x}$, $y = -3e^{-x}$, etc. This should give the set of graphs shown.



A **particular solution** (or **solution curve**) of a differential equation is a unique function that is found by giving specific values to the parameters in the general solution. The parameters in the solution are chosen so that the particular solution satisfies one or more extra requirements called **initial conditions** or **initial values**. Initial conditions are also sometimes called boundary values.

Every particular solution of a first-order differential equation is the unique solution of an appropriate **initial value problem**.

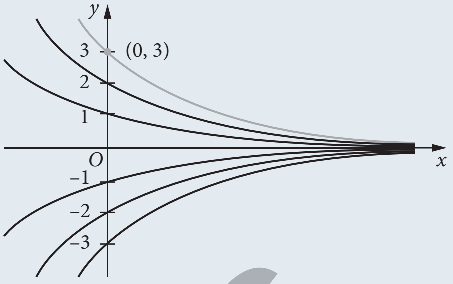
An initial value problem of a first-order differential equation: $\begin{cases} y' = f\left(t, y(t)\right) \\ y(a) = y_a \\ t \in [a,b] \end{cases}$

consists of the differential equation together with its initial condition, requiring you to determine a particular solution $y(t)$ over a specific interval of the independent variable $a \le t \le b$.

Finding the particular solution to satisfy an initial condition

Example 3

Find the particular member of the general solution $y = Ae^{-x}$, where A is a real number, that passes through the point with coordinates (0, 3).



Solution

$y = Ae^{-x}$

To satisfy the relevant initial condition, substitute the given values.

$x = 0, y = 3$: $3 = Ae^{0}$

$\therefore A = 3$

The particular solution curve that passes through the point (0, 3) is $y = 3e^{-x}$.

Finding missing parameter(s) in a trial solution

A trial solution to a differential equation is a general solution with unspecified parameters that is tested to see if it satisfies an initial condition.

Example 4

Verify by differentiation that $y = ae^{\sin x}$ is a solution of the differential equation $\frac{dy}{dx} = y\cos x$, $y(0) = 2$, for a suitable choice of the parameter a.

Solution

The required derivatives of the trial solution are calculated to prove that LHS = RHS:

$$\begin{aligned}
\text{LHS} &= \frac{dy}{dx} \\
&= \frac{d}{dx}\left(ae^{\sin x}\right) \\
&= \cos x\left(ae^{\sin x}\right) \\
&= y\cos x \\
&= \text{RHS}
\end{aligned}$$

Check if the initial condition is satisfied, $y(0) = 2$: $y(0) = ae^{\sin 0}$

$2 = a$

$\therefore\ y = 2e^{\sin x}$ is a particular solution.

Example 5

Verify by differentiation that $x = Ae^{st}$ is a solution of $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$, $x(0) = 1$ and $x'(0) = 2$, for suitable values of s.

Solution

Calculate any required derivatives of the trial solution $x = Ae^{st}$:

$$\frac{dx}{dt} = sAe^{st}$$

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) \\
&= \frac{d}{dt}\left(sAe^{st}\right) \\
&= s^2Ae^{st}
\end{aligned}$$

Substitute into the LHS of the equation given, $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$:

$$\begin{aligned}\text{LHS} &= s^2Ae^{st} - 5sAe^{st} + 6Ae^{st}\\ &= Ae^{st}\left(s^2 - 5s + 6\right)\\ &= 0\end{aligned}$$

Solve for possible values of the parameter s: $Ae^{st}(s^2 - 5s + 6) = 0$

As $Ae^{st} \neq 0$, $t \in R$, so: $(s^2 - 5s + 6) = 0$

$(s - 2)(s - 3) = 0$

$\therefore s = 2$ or $s = 3$

The solution could be $x = Ae^{2t}$ or $x = Ae^{3t}$.

Check initial conditions: $x(0) = 1$, $x = Ae^{st}$ $\therefore x(0) = Ae^0 = 1$ $\therefore A = 1$

$x'(0) = 2$, $x' = sAe^{st} = se^{st}$ $\therefore se^0 = 2$ $\therefore s = 2$

Hence $s = 2$ is the only solution that satisfies both the differential equation and the initial conditions.

$\therefore x(t) = e^{2t}$ is a particular solution.

The number of parameters in the general solution of a differential equation, which determines the number of initial conditions required to fix these parameters, depends on the **order** and the **degree** of the differential equation.

The order of a differential equation is equal to the highest order derivative of the dependent variable.
The degree of a differential equation is the highest power of that highest order derivative.

In the nth-order differential equation $\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$, the dependent variable (y) always appears in the numerator of any derivatives and the independent variable x appears in the denominator.

A **first-order first-degree differential equation** for the unknown dependent variable y is an equation that involves only the first derivative of y. All first-order first-degree differential equations for y can be expressed in the form $\frac{dy}{dx} = f\left(x, y\right)$ for a suitable choice of the function $f(x, y)$ of the independent variable x and dependent variable y.
The particular solution of a first-order first-degree differential equation requires a single **initial condition** $(x, y) = (x_0, y_0)$.

Classifying differential equations

Example 6

Classify the following differential equations according to their order and degree.

(a) $\frac{d\theta}{dt} = k\left(1 + 0.2\cos\theta\right)^2$ **(b)** $\left(\frac{dw}{dz}\right)^2 = 4\cos^2 z$ **(c)** $\frac{d^2x}{dt^2} - c\frac{dx}{dt} + kx = F$

Solution

(a) This equation defines a first-order first-degree differential equation because it involves only a first-order derivative to a power of one.

Variable θ is in the numerator of $\frac{d\theta}{dt}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

(b) This equation defines a first-order second-degree differential equation because it involves only a first-order derivative to a power of two.

Variable w is in the numerator of $\frac{dw}{dz}$, so it is the dependent variable, while z is in the denominator, so it is the independent variable.

(c) This equation defines a second-order first-degree differential equation because it involves a second-order derivative $\frac{d^2x}{dt^2}$ of the dependent variable to the power of one.

Variable x is in the numerator of $\frac{d^2x}{dt^2}$, so it is the dependent variable, while t is in the denominator, so it is the independent variable.

Note: In each example considered so far, the solution is given in explicit form, such as $y = g(x)$.

EXERCISE 14.1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

1 Verify by differentiation that the function $y = x^n$ is a solution of the differential equation $y' - \frac{n}{x}y = 0$.

2 Use technology to graph the particular solutions that correspond to the indicated values of C.

	General solution	Differential equation	C-values
(a)	$y = Cx^3$	$x\frac{dy}{dx} - 3y = 0$	$\pm1, \pm2$
(b)	$x^2 - y^2 = C$	$y\frac{dy}{dx} = x$	$\pm1, \pm2$
(c)	$x^2 + 4y^2 = C$	$4y\frac{dy}{dx} + x = 0$	1, 2, 3, 4
(d)	$xy = C$	$x\frac{dy}{dx} + y = 0$	$\pm1, \pm2$

3 Verify the general solution and then specify any parameters in this solution and state the required particular solution of the initial value problem.

(a)

General solution	Differential equation	Initial condition
$y = Ae^{-2x} + 10$	$y' = 2(10 - y)$	$y(0) = 3$

(b)

General solution	Differential equation	Initial condition
$y = Ae^{-x} + 5$	$y' = 5 - y$	$y(0) = 10$

(c)

General solution	Differential equation	Initial condition
$y = \frac{e^x}{A + e^x}$	$y' = y(1 - y)$	$y(0) = 2$

(d)

General solution	Differential equation	Initial condition
$y = \frac{5e^{10x}}{A + e^{10x}}$	$y' = 2y(5 - y)$	$y(0) = 1$

(e)

General solution	Differential equation	Initial condition
$y = Ae^{2x} - \frac{x}{2} - \frac{1}{4}$	$y' = 2y + x$	$y(0) = 1$

4 Verify the general solution and then specify any parameters in this solution to find the required particular solution.

(a)

Trial solution	Differential equation	Initial condition
$y=\frac{-x}{Ax+1}$	$x^2y'=-y^2$	$y(1)=-\frac{1}{3}$

(b)

Trial solution	Differential equation	Initial condition
$y=\frac{1}{x-A}$	$y'=-y^2$	$y(1)=-1$

(c)

Trial solution	Differential equation	Initial condition
$y=Ae^{-2x}+\frac{1}{2}$	$y'=-2y+1$	$y(0)=1$

(d)

Trial solution	Differential equation	Initial condition
$y=e^{-x}(ax+b)$	$y''+2y'+y=0$	$y(0)=2,\ y'(0)=1$

(e)

Trial solution	Differential equation	Initial condition
$y=a\sin 2x+b\cos 2x$	$\frac{d^2y}{dx^2}+4y=0$	$y(0)=1,\ y'(0)=2$

(f)

Trial solution	Differential equation	Initial condition
$y=ae^{-x}+be^{3x}$	$y''-2y'-3y=0$	$y(0)=1,\ y'(0)=1$

5 Classify each of the following differential equations in terms of its order, degree, dependent variable and independent variable.

(a) $(y')^2=x^2$ (b) $x'=x\sin(t)$ (c) $\frac{d^2x}{dt^2}+kx=0$ (d) $\frac{dy}{dx}=y+y^2$

6 Given that $y=e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2}=\frac{dy}{dx}+2y$, find the possible value(s) of k.

7 Which one of the following differential equations is satisfied by $y=\cos 2x$?

A $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4y=4\cos 2x$ B $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+4y=4\cos 2x$

C $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+4y=4\sin 2x$ D $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4y=4\sin 2x$

8 Verify by differentiation that the given function is a solution of the corresponding differential equation.

(a) $y=e^{x^n}$ is a solution of $y'-nx^{n-1}y=0$ (b) $y=x-x^{-1}$ is a solution of $xy'+y=2x$

(c) $y=\frac{x}{1+x}$ is a solution of $y'-(1-y)^2=0$ (d) $y=\frac{e^{rx}}{1+e^{rx}}$ is a solution of $y'=r(1-y)y$

(e) $y=\frac{1}{4-x^2}$ is a solution of $y'=2xy^2$

9 The amount $m(t)$ of medication remaining in the bloodstream t hours after swallowing a pill can be modelled by the differential equation $\frac{dm}{dt} = -3m + 4e^{-2t}$.

The first term on the RHS represents the rate at which the medication is absorbed from the blood into the body and the second term represents the rate at which the medicine enters the bloodstream. (This is exponential because it is rapid at first, as most of the pill dissolves, then later becomes slower when only a small amount of the pill remains.)

(a) Verify by differentiation that $m(t) = 4\left(e^{-2t} - e^{-3t}\right)$ is a solution of $\frac{dm}{dt} = -3m + 4e^{-2t}$.

(b) What is the initial amount of medication in the bloodstream?

(c) When is the amount of medication in the bloodstream at it greatest?

(d) What is the long-term amount of medication in the bloodstream?

10 The population of fish in a lake is initially 10 000. The population would increase at a rate of 20% per year, except that there is fishing quota of k fish per year taken from the lake. The population, P, after, t, years is modelled by the solution of the differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P - k \text{ with } P = 10\,000 \text{ when } t = 0.$$

Here, k is the constant number of fish removed from the lake each year due to fishing (the fishing quota).

(a) If the fishing quota is set at $k = 1000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000\left(1 + e^{\frac{t}{5}}\right)$.

(b) Alternatively, if the fishing quota is set at $k = 3000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000\left(3 - e^{\frac{t}{5}}\right)$.

More generally, you can assume an arbitrary but fixed fishing quota of k fish per year.

(c) Verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5\left(k + (2000 - k)e^{\frac{t}{5}}\right)$.

(d) Hence choose a fishing quota of k fish per year to maintain the fish population at its initial value of 10 000.

14.2 SLOPE FIELDS

Qualitative (or graphical) methods of solution

Qualitative methods are a set of graphical methods to describe the general behaviour of the solution to a differential equation without solving the equation.

Recall that $\frac{dy}{dx}$ is the slope of the curve at any point (x, y). A differential equation, such as $\frac{dy}{dx} = f(x, y)$, can be thought of as a definition of the values of the slope of the tangent to the solution curve for possible values of x and y. This enables us to sketch the graphical features of the solution. The graph showing the gradient at different points is called the **direction field** or the **slope field**.

Direction field construction on a rectangular grid

This method involves the following two steps:

1. Evaluate the derivative for a carefully selected set of points (x, y).
2. At each point (x, y), draw a short line segment of slope $\frac{dy}{dx}$.

Example 7

Construct the slope field of $\frac{dy}{dx} = xy$ on the grid:

$(0, 0)$	$(0, 1)$	$(0, 2)$	$(0, 3)$
$(1, 0)$	$(1, 1)$	$(1, 2)$	$(1, 3)$
$(2, 0)$	$(2, 1)$	$(2, 2)$	$(2, 3)$
$(3, 0)$	$(3, 1)$	$(3, 2)$	$(3, 3)$

Solution

$\frac{dy}{dx} = f(x, y) = xy$ is evaluated for each point using integer values for x and y.

For example, at the point (0, 0), $\frac{dy}{dx} = xy = 0 \times 0 = 0$.

Therefore, if the curve goes through (0, 0), its gradient at that point will be 0.

At the point (2, 3), $\frac{dy}{dx} = xy = 2 \times 3 = 6$.

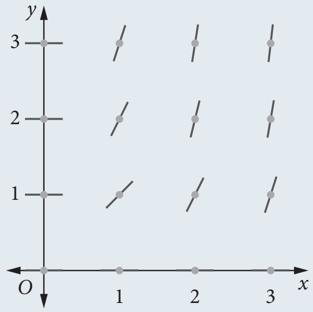
Therefore, if the curve goes through (2, 3), its gradient at that point will be 6.

All the gradients are calculated.

$f(0, 0) = 0$	$f(0, 1) = 0$	$f(0, 2) = 0$	$f(0, 3) = 0$
$f(1, 0) = 0$	$f(1, 1) = 1$	$f(1, 2) = 2$	$f(1, 3) = 3$
$f(2, 0) = 0$	$f(2, 1) = 2$	$f(2, 2) = 4$	$f(2, 3) = 6$
$f(3, 0) = 0$	$f(3, 1) = 3$	$f(3, 2) = 6$	$f(3, 3) = 9$

At each such point (x, y) on the grid, tangent segments of slope $\frac{dy}{dx} = f(x, y)$ are drawn using rise over run.

Having constructed a slope field, the short sloping lines can be used as a guide to draw smooth curves with the same gradients. These curves represent possible graphs generated by the differential equation. In some cases, more slopes may need to be drawn.

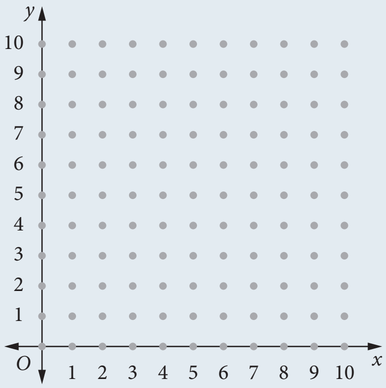


Example 8

Construct the slope field of $\frac{dy}{dx} = -2(y - 5)$ on a suitable grid for $0 \le x \le 10$ and $0 \le y \le 10$.

Solution

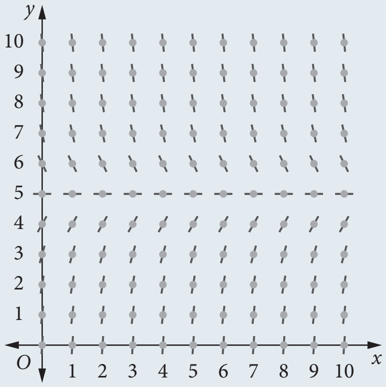
Step 1: A grid is constructed to cover the given intervals:



Step 2: At each such point (x, y) on the grid, tangent segments of the slope function are drawn. As the derivative is a function of y only, then given any specific value of y, it will be the same for all values of x. This means it only needs to be calculated for each value of y.

For $f(x, y) = -2(y - 5)$, $f(x, 0) = 10$, $f(x, 1) = 8$, $f(x, 2) = 6$,
$f(x, 3) = 4$, $f(x, 4) = 2$, $f(x, 5) = 0$, $f(x, 6) = -2$,
$f(x, 7) = -4$, $f(x, 8) = -6$, $f(x, 9) = -8$, $f(x, 10) = -10$.

This information is shown in the diagram at right.



Example 9

(a) Construct the slope field of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ for $-3 \le x \le 3$ and $-3 \le y \le 3$.

(b) Use the slope field to draw possible solutions to $\frac{dy}{dx} = -\frac{x}{y}$.

(c) Draw the specific solution if the curve passes through the point: **(i)** $(0, 2)$ **(ii)** $(2, -2)$.

(d) Suggest a possible equation of the general curve and test your answer by differentiation.

Solution

(a) **Step 1**: The differential equation is used to find the gradient at each point.

$$(-3, -3): \frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-3} = -1$$

$$(-3, -2): \frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-2} = -1.5$$

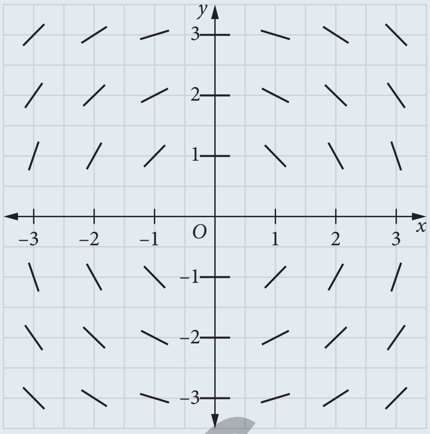
$$(-3, -1): \frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-1} = -3$$

$(-3, 0)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{0}$. This is undefined, but it may mean that if the curve goes through $(-3, 0)$, it will be vertical.

Continuing similarly for positive x values gives the following gradient values:

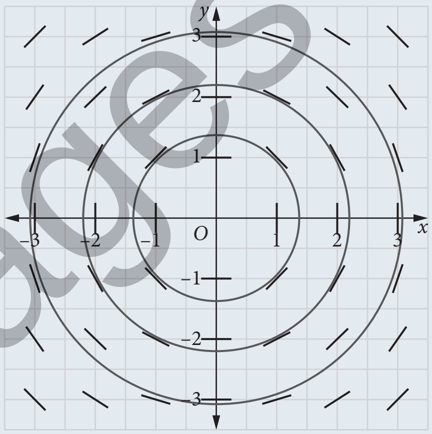
		x values						
		−3	−2	−1	0	1	2	3
y values	−3	−1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
	−2	−1.5	−1	−0.5	0	0.5	1	1.5
	−1	−3	−2	−1	0	1	2	3
	0	–	–	–	–	–	–	–
	1	3	2	1	0	−1	−2	−3
	2	1.5	1	0.5	0	−0.5	−1	−1.5
	3	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	−1

Step 2: Coordinate axes can be used to represent the slope using a short line at each point, estimating the slope using rise over run.

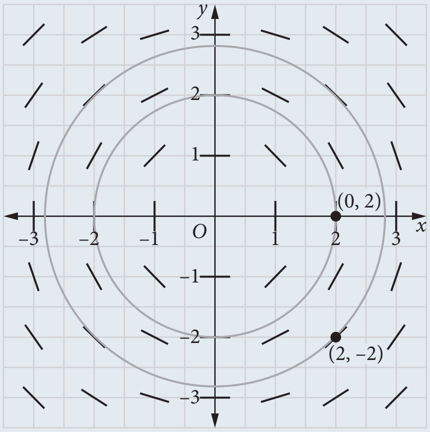


(b) The curve may be circular. Some curves can be drawn using the slopes as a guide, possibly with a compass. This diagram reinforces that the undefined slopes are at points where the curves are vertical, except for the point (0, 0). These slopes have been added to the diagram.

Circles seem to fit very well. At this point there are an infinite number of solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.



(c) A circle is drawn through each given point, as shown.



(d) The equation of a circle is $x^2 + y^2 = r^2$.

(i) In this case, $r = 2$, so the equation is $x^2 + y^2 = 4$.

If a function is required, $y = \sqrt{4 - x^2}$. The positive square root is taken because the curve goes through (0, 2).

(ii) Substitute $x = 2$, $y = -2$ in $x^2 + y^2 = r^2$.

$2^2 + (-2)^2 = r^2 \therefore r = \sqrt{8}$, so the equation is $x^2 + y^2 = 8$.

If a function is required, $y = -\sqrt{8 - x^2}$. The negative square root is taken because the curve goes through (2, −2).

The general result is tested by differentiation.

Method 1

Need to differentiate each term of $x^2 + y^2 = r^2$ with respect to x:

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(r^2\right)$$

Using the chain rule,

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dy}\left(y^2\right) \times \frac{dy}{dx} = \frac{d}{dx}\left(r^2\right)$$

Remembering that r^2 is a constant:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Method 2

Need to differentiate both sides of $y = \sqrt{r^2 - x^2}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{r^2 - x^2}\right)$$

Let $u = r^2 - x^2$, so $\frac{du}{dx} = -2x$

$$\frac{dy}{dx} = \frac{d}{du}\left(\sqrt{u}\right) \times \frac{du}{dx}$$

$$= \frac{d}{du}\left(u^{\frac{1}{2}}\right) \times \frac{du}{dx}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \times (-2x)$$

$$= -\frac{x}{\sqrt{u}}$$

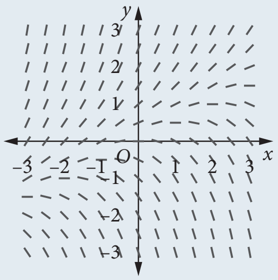
However, $y = \sqrt{r^2 - x^2} = \sqrt{u}$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

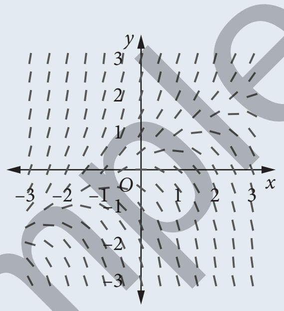
Example 10

Which of the following slope fields could have the differential equation $\frac{dy}{dx} = x + ky$ as a solution if $k > 0$?

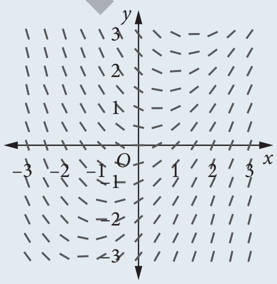
A



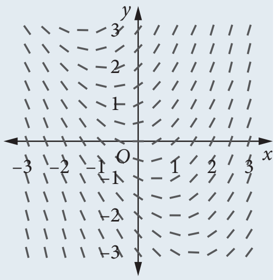
B



C



D



Solution

The behaviour of the derivative is considered for different values of x and y.

At $x = 0$ (the y-axis), $\frac{dy}{dx} = ky$, so $\frac{dy}{dx}$ will be positive where y is positive and negative where y is negative.

This eliminates C.

If $x > 0$ and $y > 0$, $\frac{dy}{dx}$ will be positive and increases as x increases. $\frac{dy}{dx}$ also increases as y increases.

This eliminates A and B.

Further analysis will show that D is consistent with $\frac{dy}{dx} = x + ky$, $k > 0$.

MAKING CONNECTIONS

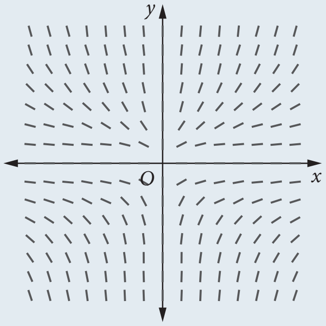
Slope fields

Use technology to construct the direction field of a differential equation.

Example 11

The slope field for a differential equation is shown:

Which of the following could be the form of the differential equation represented by this slope field, if $k > 0$?



A $\frac{dy}{dx} = kxy^2$ B $\frac{dy}{dx} = kxy^3$

C $\frac{dy}{dx} = \frac{ky^2}{x}$ D $\frac{dy}{dx} = \frac{ky^3}{x}$

Solution

Consider how the slope lines are changing and whether this is consistent with each proposed option.

By examining option A, if $x > 0$ and $y > 0$, the slope is positive and if x is constant (any horizontal row), so the slope should increase as x increases. However, if a horizontal row is examined, the slope decreases as x increases. This also rules out option B.

Alternatively, the gradient in the slope field is undefined where $x = 0$, yet the derivatives in options A and B are defined and zero where $x = 0$.

For options C and D, if $x > 0$ and $y > 0$ the slope is expected to be positive. This is consistent with the slope field. The gradients are undefined where $x = 0$, which is also consistent with the slope field.

The difference between these two options is the power of y.

If $x > 0$ and $y > 0$, the slope should increase as y increases, which can be seen by examining any vertical column.

If $x > 0$ and $y < 0$, the slope in C will be positive and increase as y increases, but the slope in D will be negative and decrease (become more negative, i.e. steeper) as y increases. By looking at the fourth quadrant, it can be seen that option C correctly predicts the behaviour of the slope.

EXERCISE 14.2 SLOPE FIELDS

1 Consider the differential equation $\frac{dy}{dx} = y - \frac{x}{2}$.

(a) Find the gradient of a solution curve at the point (3, –2), assuming the curve goes through this point.

(b) Use integer values of x and y from –3 to 3 to construct a direction field for the differential equation $\frac{dy}{dx} = y - \frac{x}{2}$.

2 Construct direction fields for the following differential equations for $-3 \le x \le 3$ and $-3 \le y \le 3$. Use integer values of x and y.

(a) $\frac{dy}{dx} = x - y$ (b) $\frac{dy}{dx} = \frac{2}{x+y}$ (c) $\frac{dy}{dx} = \frac{x^2+y^2}{10}$ (d) $\frac{dy}{dx} = \frac{x+1}{y-1}$

(e) $(x+1)\frac{dy}{dx} = xy$ (f) $x + y + \frac{dy}{dx} = 0$ (g) $\frac{dy}{dx} = \sqrt{3-y}$ (h) $\frac{dy}{dx} = (x-1)(x+2)$

(i) $\frac{dy}{dx} = y^2 - 5$ (j) $\frac{dy}{dx} = \frac{x}{2} + 1$

3 **(a)** Calculate the value of the derivative for each of the following values of y, and hence sketch the slope field of the differential equation $\frac{dy}{dx} = 1 + y^2$, using values of x from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, increasing by $\frac{\pi}{6}$.

(i) $y = 0$ **(ii)** $y = \pm 1$ **(iii)** $y = \pm 2$ **(iv)** $y = \pm 3$

(b) Verify by substitution that $y = \tan x$ is a solution of $\frac{dy}{dx} = 1 + y^2$ and sketch that particular solution of the curve on the slope field.

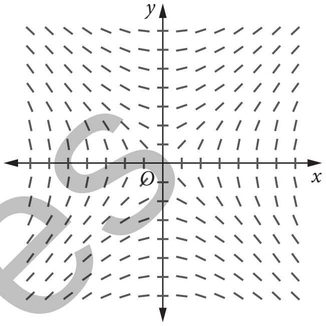
(c) State at least one point on the slope diagram which will not be on any solution of $\frac{dy}{dx} = 1 + y^2$.

4 The graph shown is the slope field of a first-order differential equation.

This differential equation could be:

A $y' = \frac{y}{x}$ **B** $y' = \frac{x}{y}$

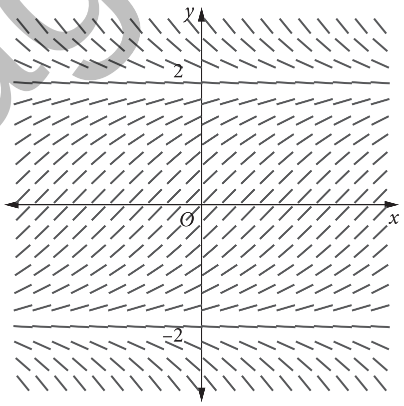
C $y' = -\frac{y}{x}$ **D** $y' = -\frac{x}{y}$



5 The slope field of $\frac{dy}{dx} = f(y)$ is shown.

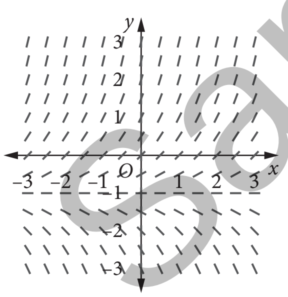
For each of the following, sketch a possible curve which is a solution to this differential equation, containing a point for which:

(a) $y > 2$
(b) $-2 < y < 2$
(c) $y < -2$

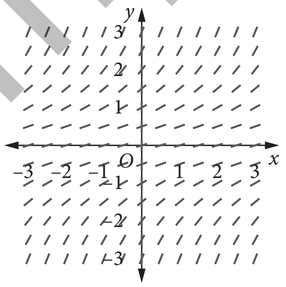


6 Which of the following slope fields does not represent a differential equation of the form $\frac{dy}{dx} = f(y)$?

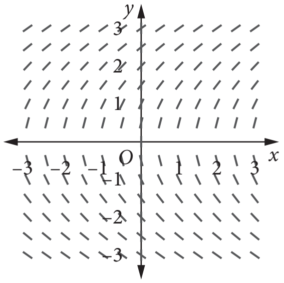
A



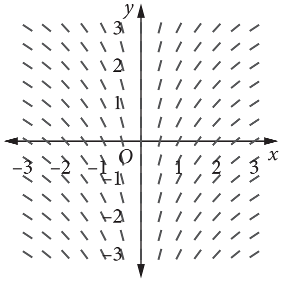
B



C

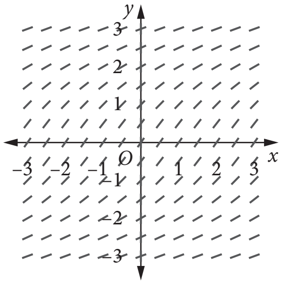


D

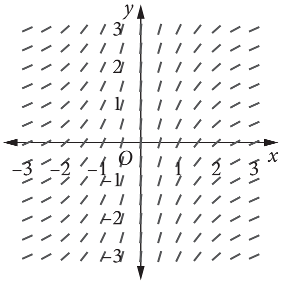


7 Which of the following slope fields represents a differential equation of the form $\frac{dy}{dx} = f(y)$?

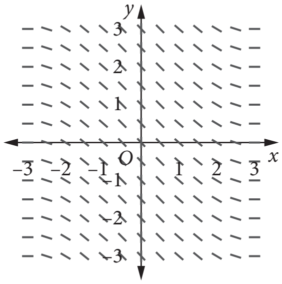
A



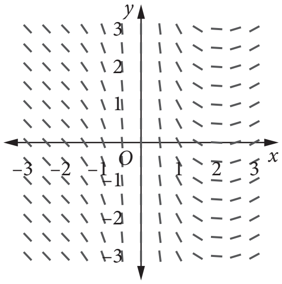
B



C



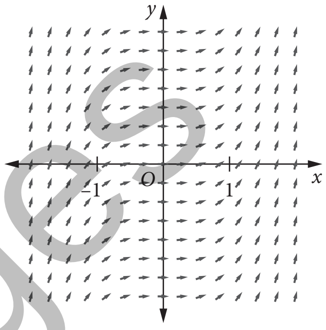
D



8 A first-order differential equation has a slope field as shown.

(a) Sketch three possible solutions for this differential equation.

(b) Which of the following first-order differential equations is consistent with the slope field shown?

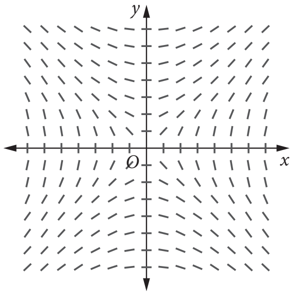


A $\frac{dy}{dx} = xy$ B $\frac{dy}{dx} = x^2$

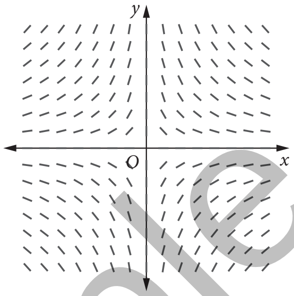
C $\frac{dy}{dx} = x^3$ D $\frac{dy}{dx} = x + y$

9 The slope field of $xy' - y = 0$ could be:

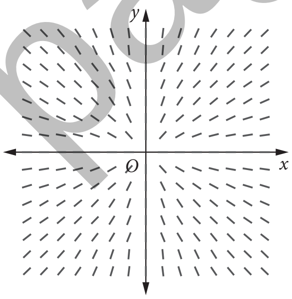
A



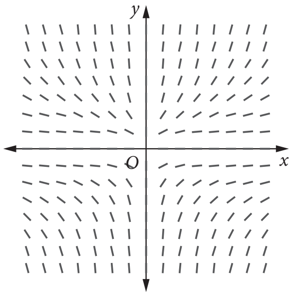
B



C



D



10 **(a)** Construct the direction field for the differential equation $\frac{dy}{dx} = x + y$, for $-3 \le x \le 3$ and $-3 \le y \le 3$, with x and y increasing in steps of 0.5.

(b) Draw some possible solutions to the differential equation $\frac{dy}{dx} = x + y$, including one that is a straight line, and including one that touches but does not cross the x-axis.

(c) Write the equation of the possible straight line solution.

(d) Verify whether the straight line represents a solution to the differential equation.

14.3 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

The previous section investigated techniques used to develop a graphical and/or numerical representation of the solution to a differential equation. However, the most convenient solution (when it is available) is an expression for the dependent variable as an explicit function of the independent variable or as an implicit relation between the dependent and independent variables. This section covers a number of powerful techniques capable of determining these analytical solutions.

Solving $\frac{dy}{dx} = f(x)$ given $y(a) = y_a$

Begin with the directly integrable case $\frac{dy}{dx} = f(x)$ where y is an unknown function of x and $f(x)$ is a given function of x. Wherever $f(x)$ is a continuous function over an interval, $y(x)$ can be determined by using an appropriate integration technique.

Solving a directly integrable first-order differential equation

Example 12

A one-parameter family of curves, $f(x, y) = c$, has the property that the gradient of any member of the family at a point is three times the square of the x-coordinate at the point.

(a) What is the equation of the family of functions?

(b) Plot a few members of the family.

(c) What is the equation of the particular member of the family that passes through the point (1, 6)?

Solution

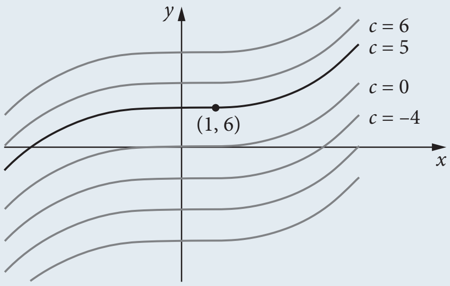
(a) Possible gradient function: $\frac{dy}{dx} = 3x^2$.

Both sides of the model are integrated with respect to the independent variable: $\int \frac{dy}{dx}dx = \int 3x^2 dx$

Equation of the family of functions: $y = x^3 + c$

Note: By the chain rule, $\int \frac{dy}{dx}dx = \int dy$. A constant of integration is required when integrating.

(b)



(c) The coordinates of the given point (1, 6) are substituted into the equation of the family of functions:

$6 = 1^3 + c$

$c = 5$

Solution: $y = x^3 + 5$

Example 13

The slope of the tangent line to an unknown curve is $\frac{1}{1+x^2}$. The y-intercept of the curve is (0, 1). Find the equation of the curve.

Solution

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\int \frac{dy}{dx}dx = \int \frac{dx}{1+x^2}$$

$$y = \tan^{-1} x + c$$

$$y(1) = 0 : 1 = \tan^{-1} 0 + c$$

$$c = 1$$

$$\therefore\ y = \tan^{-1} x + 1$$

Finding the particular solution of a directly integrable problem $\frac{dy}{dx} = f(x)$, $y(a) = y_a$ involves the following two-step procedure.

1 Integrate with respect to the independent variable.

That is, $y = F(x) + c$ where $F(x) = \int f(x)dx$ and c is the constant of integration.

2 Use the initial condition to solve the constant of integration.

That is, $(x, y) = (a, y_a) \therefore y_a = F(a) + c$.

Therefore, $y = F(x) + y_a - F(a)$.

Rearranging the previous equation, the final result is $y(x) = y_a + F(x) - F(a)$.

Example 14

Given $y' = \frac{1}{\sqrt{1-x^2}}$ and $y(0) = 2$, find the solution to the differential equation.

Solution

$$y' = \frac{1}{\sqrt{1-x^2}}$$

$$y = \int \frac{dx}{\sqrt{1-x^2}}$$

$$y = \sin^{-1} x + C$$

$$y(0) = 2 : 2 = \sin^{-1} 0 + C$$

$$C = 2$$

$$\therefore\ y = \sin^{-1} x + 2$$

Example 15

Find the solution to the following differential equations:

(a) $\frac{dy}{dx} = \sin x$, given $y(0) = 1$ **(b)** $\frac{dy}{dx} = 2 - \cos x$, given $y(0) = 2$.

(c) Hence, use parts **(a)** and **(b)** to solve the second-order differential equation $\frac{d^2y}{dx^2} = \sin x$, given that $\frac{dy}{dx} = 1$ and $y = 2$ where $x = 0$.

Solution

(a)

$$\frac{dy}{dx} = \sin x$$

$$y = \int \sin x\, dx$$

$$y = -\cos x + C$$

$y(0) = 1$:

$$1 = -1 + C$$

$$C = 2$$

$$\therefore\ y = 2 - \cos x$$

(b)

$$\frac{dy}{dx} = 2 - \cos x$$

$$y = \int (2 - \cos x)dx$$

$$y = 2x - \sin x + c$$

$y(0) = 2$:

$$2 = 2 \times 0 - \sin 0 + c$$

$$c = 2$$

$$\therefore\ y = 2x - \sin x + 2$$

(c) Now: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

If $\frac{d^2y}{dx^2} = \sin x$ then the solution of this differential equation will give an equation of the form $\frac{dy}{dx} = f(x)$.

From part **(a)** you can obtain that $\frac{dy}{dx} = 2 - \cos x$.

From part **(b)** you can obtain the solution of this equation, it is $y = 2x - \sin x + 2$.

Hence the solution to $\frac{d^2y}{dx^2} = \sin x$ with the given initial conditions is $y = 2x - \sin x + 2$.

Example 16

Solve the differential equation $\frac{dy}{dx} = \frac{4}{x^2 - 4}$, given that $y = -\ln 5$ when $x = 3$.

It is also given that $\frac{1}{x^2-4} = \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)$.

Solution

$$\frac{4}{x^2-4} = \frac{1}{x-2} - \frac{1}{x+2}$$

$$\frac{dy}{dx} = \frac{1}{x-2} - \frac{1}{x+2}$$

$$y = \int\left(\frac{1}{x-2} - \frac{1}{x+2}\right)dx$$

$$y = \ln|x-2| - \ln|x+2| + C$$

$$(3, -\ln 5): -\ln 5 = \ln 1 - \ln 5 + C$$

$$C = 0 \text{ so } y = \ln\left|\frac{x-2}{x+2}\right|$$

Example 17

Solve the differential equation $\frac{dy}{dx} = \frac{x^2+2}{x^2+1}$, given that $y(0) = 2$.

Solution

$$\frac{x^2+2}{x^2+1} = \frac{1+x^2+1}{x^2+1} = 1 + \frac{1}{x^2+1}$$

$$\text{Thus: } \frac{dy}{dx} = 1 + \frac{1}{x^2+1}$$

$$y = \int\left(1 + \frac{1}{x^2+1}\right)dx$$

$$y = x + \tan^{-1}x + C$$

$$(0,2): 2 = 0 + \tan^{-1}0 + C$$

$$C = 2 \text{ so } y = x + \tan^{-1}x + 2$$

EXERCISE 14.3 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

1 Write the general solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x - 1$ **(b)** $f'(x) = x^2\sqrt{x}$ **(c)** $y'(x) = 2\cos 2x$ **(d)** $y'(x) = 2\cos^2 x$

(e) $\frac{dz}{dt} = \frac{1}{t^2 + 4}$ **(f)** $\frac{dz}{dt} = \frac{t}{t^2 + 4}$ **(g)** $\frac{dx}{d\theta} = \sin^2\theta + \cos^2\theta$ **(h)** $f'(x) = 1 - e^{\frac{-x}{2}}$

2 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x^3 - x + 1$, given that $y = 2$ where $x = 1$

(b) $f'(x) = \sin x(1 - \cos x)$, given that $f\left(\frac{\pi}{3}\right) = -1$

(c) $x'(t) = \sqrt{1+t}$, given that $x(0) = 3$

(d) $\frac{dV}{dt} = te^{t^2}$, given that $V = 2$ when $t = 0$

(e) $\frac{dx}{d\theta} = \frac{\sin\theta}{2 + \cos\theta}$, given that $x = 1$ where $\theta = \pi$

(f) $\frac{du}{dx} = \frac{x^2}{x^2 + 1}$, given that $u = 5$ where $x = 0$

(g) $\frac{dy}{dt} = t\sqrt{t^2 - 9}$, $t \geq 3$, given that $y = 1$ when $t = 3$

3 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$, $y(0) = 1$

(b) $\frac{dx}{dt} = \frac{t}{t^2 + 1}$, $x = 1$ where $t = 0$

(c) $\frac{dx}{dy} = \frac{y}{2y - 2}$, given that $x = 1$ where $y = 2$

(d) $\frac{dz}{dt} = \frac{t}{\sqrt{1 - t}}$, $z(-2) = 0$

4 Solve the following differential equations for the initial conditions given.

(a) $\frac{dy}{dx} = \frac{x + 1}{x^2}$ if $x > 0$ and $y = 1$ when $x = 1$.

(b) $\frac{dy}{dx} = x\sqrt{1 + x^2}$, given that $y = \frac{4}{3}$ when $x = 0$.

(c) $\frac{dy}{dx} = 1 + \sin x + \sin^2 x$, given that $y = 2$ when $x = 0$.

(d) $\frac{dy}{dx} = \frac{2}{1 - x^2}$, given that $y = 0$ when $x = 0$ and that $\frac{1}{1 - x^2} = \frac{1}{2}\left(\frac{1}{1 + x} + \frac{1}{1 - x}\right)$.

5 The solution to $\frac{dy}{dx} = x$ with initial condition $y(0) = -1$ is:

A always concave down **B** always concave up **C** always decreasing **D** always increasing

6 Consider $\frac{d}{dx} = (xe)^x$

(a) Show that the differential expression is equal to $e^x + xe^x$.

(b) Hence find $\int xe^x \, dx$.

(c) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x$, given $y(0) = -1$.

(d) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x - e^x$, given $y(0) = -2$.

(e) Hence find the particular solution for the second-order differential equation $\frac{d^2y}{dx^2} = xe^x$, given that $\frac{dy}{dx} = -1$ and $y = -2$ where $x = 0$.

7 Find the particular solution of the differential equation $y' = \frac{1}{x^2 + 1}$ if $y(0) = 1$.

8 Consider $\frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2 + 1\right)\right)$

(a) Show that this differential expression equals $\tan^{-1}x + 1$.

(b) Using **(a)**, find the particular solution of the differential equation $\frac{dy}{dx} = \tan^{-1}x + 1$ if $y(0) = 0$.

9 Consider the expression $\frac{1}{2}\left(e^x - e^{-x}\right)$.

(a) If $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with initial condition $y(0) = 1$, find y.

(b) If $\frac{dz}{dx} = \frac{1}{2}\left(e^x + e^{-x}\right)$ with initial condition $z(0) = 0$, find z.

(c) Hence show that if $\frac{d^2y}{dx^2} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with $y(0) = 0$ and $y'(0) = 1$, then $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ is a particular solution of this equation.

10 A simple economic model for the rate of change $\frac{dP}{dt}$ of the price, P, of a product with respect to time, t, states that $\frac{dP}{dt}$ varies directly with the difference between the demand, D, and the supply, S, of the product. This can be written as $\frac{dP}{dt} = r(D - S)$, $P(0) = 1$ with r being a positive constant.

Let us assume the supply of widgets is given by $S(t) = 2 + \sin^2\left(\frac{\pi t}{6}\right)$ and the demand for widgets by $D(t) = 3 + \cos^2\left(\frac{\pi t}{6}\right)$, where t is the time in months since the start of the year.

(a) Express $P(t)$ in terms of r and t. The widget price after three months is \$4.

(b) Sketch the widget price over the course of the year.

11 An oil tanker hits a reef and spills oil into the sea. The oil spills from the tanker at a rate of $\frac{10^6 t}{t^4 + 16}$ litres day^{-1}, where t is the number of days since the tanker first hit the reef.

It is known that $\int \frac{t}{t^4 + 16}\,dt = \frac{1}{8}\arctan\left(\frac{t^2}{4}\right) + C$.

(a) If V litres is the volume of oil spilled into the sea in the first T days, find V in terms of T.

The local newspaper report stated, 'It is expected that eventually 300 000 litres of oil will spill into the sea.'

(b) Determine whether the newspaper report is in agreement with the model above.

Challenging

12 An irregularly shaped tank 10 metres high, holds 100 cubic metres of oil when full. The volume, $V(h)$ cubic metres, of oil in a partially filled tank, where the depth of oil is h metres, is given by $V(h) = Ah\sin^{-1}\left(\frac{h}{10}\right)$ where A is a real number.

(a) Find the exact value of A.

The tank is initially empty. Oil is then pumped into the tank at a constant rate of 2 cubic metres per minute.

(b) Find, in terms of h, an expression for the rate at which the volume of the oil is increasing, in metres per minute, where the depth is h metres.

(c) Find an expression for the derivative $\frac{dt}{dh}$ in terms of h.

(d) Hence find an expression for t in terms of h.

(e) Hence find the time t (in minutes) for the depth of oil to reach $5\sqrt{3}$ metres. Express your answer in the form $t = \frac{a\sqrt{b}}{c}$ minutes for a suitable choice of the natural numbers a, b, c.

13 A rocket has a mass 40 000 kg plus an initial mass m_F kg of fuel. At the initial time $t = 0$ seconds, the rocket is launched vertically from ground level. Ignoring the variation of gravity with height h, the altitude of the rocket is then governed by $\frac{d^2h}{dt^2} = -\frac{49}{5} + \frac{780\,000}{m_F - 200(t - 200)}$, $0 \le t \le 200$, starting with $\frac{dh}{dt} = 0$ and $h = 0$, when $t = 0$.

(a) Find $\frac{dh}{dt}$ (assume $0 \le t \le 200$).

(b) Given that

$$\frac{d}{dt}\left(3900t - \frac{49}{10}t^2 + \frac{39}{2}\left(m_F - 200(t - 200)\right)\log_e\left|\frac{m_F - 200(t - 200)}{40\,000 + m_F}\right|\right)$$

$$= -\frac{49t}{5} + 3900\log_e\left|\frac{m_F + 40\,000}{m_F - 200(t - 200)}\right|,$$ find $h(t)$.

Burnout occurs at time $t_b = \dfrac{m_F}{200}$, when all the fuel has been expended.

(c) Find the altitude h_b of the rocket at the time of burnout.

14.4 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\dfrac{dy}{dx} = g(y)$

Models that describe systems exhibiting growth or decay have the form $\dfrac{dy}{dx} = g(y)$. In other words, the rate of growth $\dfrac{dy}{dx}$ in the dependent variable y is a function $g(y)$ of the variable y alone. The function $g(y)$ on the right-hand side of such a differential equation has no explicit dependence on the independent variable x.

To solve $\dfrac{dy}{dx} = g(y)$ given that $y(0) = y_0$, consideration has to be given to two separate cases.

Case I: $g(y_0) = 0$

If $g(y_0) = 0$, then $y(x) = y_0$. In other words, the solution $y(x) = y_0$ is not dependent on the independent variable, so it is variously called the stationary solution, the steady state solution, or the equilibrium solution, y_s.

Case II: $g(y_0) \neq 0$

The model $\dfrac{dy}{dx} = g(y)$ can be transformed to a directly integrable form $\dfrac{1}{\left(\dfrac{dy}{dx}\right)} \equiv \dfrac{dx}{dy} = \dfrac{1}{g(y)}$ by taking the reciprocal of both sides, wherever $g(y_0) \neq 0$.

The solution of this new equation gives $x(y)$, so the variable x is now a function of y. In other words, the roles of the dependent and independent variables in the original equation $\dfrac{dy}{dx} = g(y)$ have been exchanged in the transformed equation $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$.

Recall that exchanging the roles of dependent and independent variables in any relationship, such as $y = f(x)$, requires the existence of an appropriate inverse function f^{-1}. In other words, $y = f(x) \Leftrightarrow x = f^{-1}(y)$, provided f is a one-to-one function on an interval, which requires $f(x) \neq 0$ inside that interval. In this problem, $\dfrac{dy}{dx} = g(y)$ defines y as an invertible (i.e. one-to-one) function of x on an interval, wherever $g(y) \neq 0$ for any value of the variable y in the said solution. Therefore, the solution $x = g(y)$ of $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$ can always be inverted to give the solution $y = g^{-1}(x)$ of the original problem $\dfrac{dy}{dx} = g(y)$ on any interval of y where $g(y) \neq 0$.

A number of these kinds of differential equations will be investigated.

To find the particular solution of $\dfrac{dy}{dx} = g(y)$, $y(a) = y_a$, $g(y_a) \neq 0$, requires the following five-step procedure:

1. Take the reciprocal of both sides of $\dfrac{dy}{dx} = g(y)$ to obtain $\dfrac{dx}{dy} = \dfrac{1}{g(y)}$ with $x = a$ where $y = y_a$.
2. Integrate both sides of the equation with respect to the new independent variable y:
   $$\int \frac{dx}{dy}\,dy = \int \frac{1}{g(y)}\,dy$$
   $$x + c = G(y) \qquad \text{where } G'(y) = \frac{1}{g(y)} \text{ and } c \text{ is a constant of integration.}$$
3. Where possible, invert the equation from previous step to find the general solution for the original dependent variable y in terms of the original independent variable x:
   $x + c = G(y)$
   $\therefore\ y = G^{-1}(x + c)$
4. Substitute the initial condition $(x, y) = (a, y_a)$ into the equation obtained in the previous step to evaluate the constant of integration c: $\therefore\ y_a = G^{-1}(a + c)$.
5. Substitute the value of the constant into the general solution obtained in the previous step to obtain the particular solution.

Example 18

Find the solution of $\frac{dy}{dx} = 2y$, given that where $x = 0$, $y = 3$.

Solution

Take the reciprocal of both sides of the equation:

$$\frac{dx}{dy} = \frac{1}{2y}$$

Integrate with respect to y:

$$x + c = \frac{1}{2}\log_e |y|$$

Logarithms are only defined for positive quantities, so the absolute value sign is used.

Substitute known values of x and y,

$x = 0$, $y = 3$:

$$0 + c = \frac{1}{2}\log_e 3$$

Simplify and express with y as a function of x:

$$x + \frac{1}{2}\log_e 3 = \frac{1}{2}\log_e |y|$$

$$2x = \log_e |y| - \log_e 3$$

$$2x = \log_e \frac{|y|}{3}$$

$$\frac{|y|}{3} = e^{2x}$$

$$|y| = 3e^{2x}$$

This gives two solutions.

If $y > 0$, $|y| = y = 3e^{2x}$

If $y < 0$, $|y| = -y = 3e^{2x}$ $\therefore\ y = -3e^{2x}$

Since $y > 0$ when $x = 0$ the solution is: $y = 3e^{2x}$

Example 19

Find the particular solution of $\frac{dy}{dt} = -r(y - s)$, given that $y(0) = y_0$ with $r, s > 0$ positive constants.

Solution

Take the reciprocal of both sides and transpose $-r$:

$$-r\frac{dt}{dy} = \frac{1}{y - s}$$

Integrate both sides with respect to the original dependent variable y:

$$-r\int \frac{dt}{dy}\,dy = \int \frac{1}{y - s}\,dy$$

$$-rt + c = \log_e |y - s|$$

Rearrange to find the general solution:

$$|y - s| = e^c e^{-rt}$$

$$\therefore\ y - s = Ae^{-rt}$$

$$y = s + Ae^{-rt}, \quad \text{where } A = \pm e^c.$$

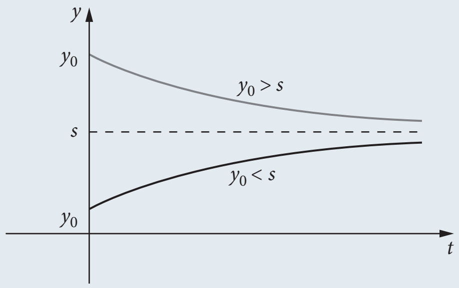
To find the particular solution, substitute initial condition $y(0) = y_0$:

$$y_0 - s = Ae^0$$

$$\therefore\ A = y_0 - s$$

$$\therefore\ y = s + (y_0 - s)e^{-rt}$$

It appears that this is a single solution. However, this solution will behave differently depending on the relative size of the initial condition $y(0) = y_0$ and the steady state solution s, as shown.



In the case that $(y_0 - s) < 0$, the solution $y = s + (y_0 - s)e^{-rt}$ grows as time increases.

In the case that $(y_0 - s) > 0$, the solution $y = s + (y_0 - s)e^{-rt}$ decays as time increases.

Example **19** above is related to the model of uninhibited exponential growth/decay, $\frac{dy}{dt} = ry$, which for $y(0) = y_0$ has the general solution $y = y_0 e^{rx}$.

Assuming $r > 0$, the differential equation $\frac{dy}{dt} = -r\left(y - y_s\right)$, $y(0) = y_0$ is a model of inhibited growth for $y_0 < y_s$ and inhibited decay for $y_0 > y_s$. The rate of growth (or decay) in the dependent variable is called *inhibited* because the rate falls to zero as the dependent variable approaches its equilibrium value y_s.

The solution curve of an inhibited growth (or decay) problem is $y(t) = y_s + (y_0 - y_s)e^{-rt}$.

Example 20

(a) Show that $\frac{2}{(1-y)(1+y)} = \frac{1}{1-y} + \frac{1}{1+y}$.

(b) Find the general solution of $\frac{dy}{dx} = \left(1-y\right)\left(1+y\right)$, given that $y(0) = y_0$.

Solution

(a)

$$\begin{aligned}\text{RHS} &= \frac{1}{1-y} + \frac{1}{1+y} \\ &= \frac{1+y+1-y}{(1-y)(1+y)} \\ &= \frac{2}{(1-y)(1+y)} = \text{LHS}\end{aligned}$$

(b) Take the reciprocal of both sides of the equation:

$$\frac{dy}{dx} = \left(1-y\right)\left(1+y\right)$$

$$\frac{dx}{dy} = \frac{1}{\left(1-y\right)\left(1+y\right)}$$

Both sides of the equation are integrated with respect to y:

$$\int \frac{dx}{dy}\,dy = \int \frac{1}{\left(1-y\right)\left(1+y\right)}\,dy$$

The result from **(a)** is used, $\frac{1}{(1-y)(1+y)} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$:

$$\begin{aligned}x &= \frac{1}{2}\int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy \\ 2x &= \int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy \\ 2x + C &= \left(-\ln\left|1-y\right| + \ln\left|1+y\right|\right) \\ 2x + C &= \ln\left|\frac{1+y}{1-y}\right|\end{aligned}$$

Note: Add the constant of integration to the side with the original independent variable.

Rearrange the equation from the previous step to find the general solution for the original dependent variable in terms of the original independent variable:

$$e^{2x+C} = \left|\frac{1+y}{1-y}\right|$$

$$\left|\frac{1+y}{1-y}\right| = e^{2x}e^{C}$$

$$\frac{1+y}{1-y} = \pm e^{2x}e^{C}$$

Let $A = \pm e^{C}$:

$$\frac{1+y}{1-y} = Ae^{2x}$$

$$1+y = Ae^{2x} - yAe^{2x}$$

$$y\left(1+Ae^{2x}\right) = Ae^{2x} - 1$$

$$y = \frac{Ae^{2x}-1}{1+Ae^{2x}}$$

The initial condition is substituted to find the particular solution, $y(0) = y_0$:

$$y_0 = \frac{A-1}{1+A}$$

$$y_0 + Ay_0 = A - 1$$

$$y_0 + 1 = A\left(1-y_0\right)$$

$$A = \frac{1+y_0}{1-y_0}$$

Hence the solution is:

$$y = \frac{\left(\frac{1+y_0}{1-y_0}\right)e^{2x} - 1}{1+\left(\frac{1+y_0}{1-y_0}\right)e^{2x}}$$

This may also be rearranged to give:

$$y = \frac{1-\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}{1+\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}$$

The equations solved for exponential growth and decay in the Mathematics Advanced course, $\frac{dN}{dt} = kN$ and $\frac{dQ}{dt} = -kQ$, are examples of the solution of this type of differential equation.

The solution of the quadratic growth rate model $\frac{dy}{dx} = (1-y)(1+y)$, $y(0) = y_0$ is:

$$y = \frac{1-\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}{1+\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}.$$

EXERCISE 14.4 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

1 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -y,\ y(0) = 1$ **(b)** $\frac{dy}{dx} = 2y,\ y(0) = -1$

(c) $\frac{dy}{dx} = -2y,\ y(0) = -5$ **(d)** $\frac{dy}{dx} = 2y,\ y(0) = 3$

2 In each case, find the equation of the solution curve and then sketch its graph.

(a) $\frac{dy}{dx} = -2(y-3),\ y(0) = 8$ **(b)** $\frac{dy}{dx} = -2\left[y(x)-8\right],\ y(0) = 3$

(c) $\frac{dy}{dx} = -4(y+3),\ y(0) = 5$ **(d)** $\frac{dy}{dx} = -4(y-3),\ y(0) = -5$

3 Given that $\frac{dy}{dx} = \cos^2 y$ and that $y = \frac{\pi}{4}$ at $x = 0$, then which of the following is true?

A $y = \frac{1}{2}y + \frac{1}{4}\sin 2y$ **B** $x = \tan\left(y + \frac{\pi}{4}\right)$ **C** $y = \tan^{-1}(x+1)$ **D** $y = \tan^{-1}\left(x - \frac{\pi}{4}\right)$

4 Which of the following is a particular solution to the differential equation $2y - 3\frac{dy}{dx} = 1$?

A $y = -\frac{1}{3}$ **B** $y = -e^{\frac{1}{3}(2x-1)}$ **C** $y = \frac{1}{2}$ **D** $y = 3e^{\frac{2}{3}x}$

5 Find the general solution to $\frac{dx}{dt} = \sec x$, given that $x = \frac{\pi}{6}$ when $t = \frac{1}{2}$.

6 Consider the expression $\frac{4}{(2-y)(2+y)}$.

(a) Show that this expression is equal to $\frac{4}{(2-y)(2+y)} = \frac{1}{2-y} + \frac{1}{2+y}$.

(b) Find the general solution to $\frac{dy}{dx} = (2-y)(2+y)$, given that $y(0) = 1$.

7 Find the solution to the differential equation $\frac{dy}{dx} = \sqrt{16 - 3y^2}$ that passes through the point (0, 2).

8 Solve the differential equation $\frac{dy}{dx} = 10e^{-4y}$, given that $y = 0$ when $x = 0$.

9 Solve the differential equation $\frac{dy}{dx} = 20e^{5y}$, given that $y = 0$ when $x = 0$.

10 Find the equation of the curve in the Cartesian plane that passes through the point $\left(\frac{\pi}{3}, 1\right)$ and whose gradient at each point is equal to $\sqrt{1 - y^2}$.

11 Given $\frac{dx}{dt} = (3-x)^2$, and $x = 2$ when $t = 0$, find x as a function of t.

12 In Biology and Ecology, the term *desiccation* refers to the drying out (i.e. the loss of water) of the cells of a living organism. Most cells are made mostly of water. Assume that the desiccation of a cell is modelled by the solution of the following differential equation: $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, where V is the volume of the water in the cell, t is time and k is an appropriate constant of proportionality.

During the intense heat of the Australian summer, the cells of a newly fallen eucalyptus leaf still contain water, but the leaf loses this water rapidly through the process of desiccation. Suppose that each leaf cell initially contains 8 μm^3 of water, but 4 hours later each cell has only 1 μm^3 of water.

(a) Find the particular solution of $\frac{dV}{dt} = -kV^{\frac{2}{3}}$, $V(0) = 8\,\mu m^3$.

(b) Find the time taken for the cells to lose all of their water (assuming that the environmental conditions don't change over this time).

13 The pressure of the atmosphere, P kilopascals (kPa), decreases according to the height, h kilometres, above sea level. The rate of change of the pressure with respect to the height above sea level is proportional to the pressure at that height.

(a) Write a differential equation to describe this situation.
(b) The pressure at sea level is 101.3 kPa and it is approximately 37.3 kPa at a height of 5 km. Solve the differential equation to find P as a function of h.
(c) Estimate the air pressure at the top of Mount Everest, which is about 9 km high.

14 In an electric circuit, a capacitor, of capacitance C, charged to a potential difference, E, is discharged through a resistance, R. If q is the charge on the capacitor at time t, then $\frac{dq}{dt} = -\frac{q}{RC}$ is the differential equation describing this situation. If initially $q = EC$, then find the solution of this equation (that is, q as a function of t).

15 Newton's law of cooling states that 'the cooling rate of a body is proportional to the difference between the temperature of the body and that of the surrounding medium.' This may be written as $\frac{dT}{dt} = -k(T - M)$, where T is the temperature at any time, t, and M is the temperature of the surrounding medium (a constant).

A pot of soup is cooked at 100°C. To cool the soup, it is placed in a room where the temperature is 20°C. After 20 minutes the temperature of the soup has dropped to 70°C.

(a) Find the general solution of the differential equation $\frac{dT}{dt} = -k(T - 20)$.
(b) Find the value of k.
(c) How much time will it take the pot of soup to cool to 25°C?

Challenging

16 Two pieces of information are needed for the safe medicinal use of a drug:

(i) dosage level required to achieve the desired effect **(ii)** time interval between doses.

The body deals with a drug by first absorbing it into body tissues, then excreting it via body organs. After being absorbed, the process of drug excretion usually begins as blood takes the drug from body tisses and flows through the kidneys. Assume that the kidney's excretion rate $\frac{dy}{dt}$ is proportional to the amount of the drug, y, in the body tissues, so that $\frac{dy}{dt} = -ky$, $k > 0$.

Suppose that an initial dose, D, of a drug is given at time $t = 0$, and that this dosage is repeated at intervals of T minutes.

(a) Show that the amount of the initial dose remaining when the second dose is given is De^{-kT}.
(b) Find an expression for the amount of drug in the body just after the nth dose has been given.
(c) Show that the maximum amount of the drug that can build up in the body from a succession of doses of quantity D is $\frac{D}{1 - e^{-kt}}$. This is called the saturation level y_s and is the desired dosage level for safe maximum effectiveness.
(d) The half-life for the excretion of a drug is the time for the quantity of a drug in the body to reduce from D to $\frac{D}{2}$, when no further dose is given. If the half-life is h, show that $k = \frac{\log_e 2}{h}$.

14.5 HARDER EXPONENTIAL GROWTH AND DECAY

This is an extension of the work on exponential growth and decay that you studied in the Mathematics Advanced course, where you considered the equation $\frac{dN}{dt} = kN$. That solution was found to be $N = Ae^{kt}$, where A is the initial value of N. It is a useful exercise to derive this result by integration.

Example 21

Show by integration that the solution to $\frac{dN}{dt} = kN$ is $N = Ae^{kt}$, where A is the value of N when $t = 0$.

Solution

$$\frac{dN}{dt} = kN$$

Take the reciprocal of both sides: $\frac{dt}{dN} = \frac{1}{kN}$

Integrate with respect to N: $t = \int \frac{1}{kN}\,dN$

$$t = \frac{1}{k}\log_e N + C$$

When $t = 0$, $N = A$: $0 = \frac{1}{k}\log_e A + C$

$$C = -\frac{1}{k}\log_e A$$

$$\therefore t = \frac{1}{k}\log_e N - \frac{1}{k}\log_e A$$

$$t = \frac{1}{k}\log_e \frac{N}{A}$$

$$kt = \log_e \frac{N}{A}$$

$$\frac{N}{A} = e^{kt}$$

$$N = Ae^{kt}$$

Solution of $\frac{dN}{dt} = k(N - P)$ where k and P are constants

$\frac{dN}{dt} = k(N - P)$ means that the rate of change of N is proportional to the excess of N over a fixed quantity P. This can be applied to several real-life physical processes.

Given $\frac{dN}{dt} = k(N - P)$, where k and P are constants, it is easy to show that a solution of this equation is $N = P$.

If $N = P$, then $\frac{dN}{dt} = k(P - P) = 0$; because P is a constant, differentiating the equation $N = P$ with respect to t also gives $\frac{dN}{dt} = 0$.

It can also be shown by substitution that $N = P + Ae^{kt}$ is a solution to $\frac{dN}{dt} = k(N - P)$:

$$\text{LHS} = \frac{dN}{dt} = Ake^{kt}$$

$$\text{RHS} = k(N - P) = k(P + Ae^{kt} - P) = Ake^{kt} = \text{LHS}$$

It is important to be able to derive this result by integration.

$$\frac{dN}{dt} = k(N - P)$$

Reciprocal of both sides: $\frac{dt}{dN} = \frac{1}{k(N - P)}, \quad N \neq P$

Integrate with respect to N: $t = \frac{1}{k}\int \frac{1}{N-P}\,dN$

$$t = \frac{1}{k}\log_e(N-P) + C,\ \ N > P$$

$$k(t-C) = \log_e(N-P)$$

$$N - P = e^{k(t-C)}$$

$$N - P = e^{kt} \times e^{-kC}$$

Let $A = e^{-kC}$ (a constant): $N - P = Ae^{kt}$

$$N = P + Ae^{kt}$$

If $\frac{dN}{dt} = k(N-P)$, then $N = P + Ae^{kt}$

If $k < 0$, $Ae^{kt} \to 0$ as $t \to \infty$ and hence $N \to P$ from above.

Example 22

N is increasing according to the equation $\frac{dN}{dt} = 0.4(N-50)$. If $N = 60$ when $t = 0$:

- **(a)** show that $N = 50 + Ae^{0.4t}$ is a solution to this equation, where A is a constant
- **(b)** calculate the value of N when $t = 20$.

Solution

(a) Differentiate $N = 50 + Ae^{0.4t}$: $\frac{dN}{dt} = 0.4Ae^{0.4t}$

Rewrite $N = 50 + Ae^{0.4t}$: $Ae^{0.4t} = N - 50$

Substitute into $\frac{dN}{dt} = 0.4Ae^{0.4t}$: $\frac{dN}{dt} = 0.4(N-50)$

Hence $N = 50 + Ae^{0.4t}$ is a solution to the equation $\frac{dN}{dt} = 0.4(N-50)$.

(b) At $t = 0$, $N = 60$: $60 = 50 + A$

$$A = 10$$

$$\therefore\ N = 50 + 10e^{0.4t}$$

At $t = 20$: $N = 50 + 10e^{8}$

$$N \approx 29\,860$$

Example 23

The mass, M, of a particular southern right whale is modelled as $M = 55 - 54e^{-kt}$, where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

- **(a)** Show that the rate of growth of the whale's mass is given by the differential equation $\frac{dM}{dt} = k(55 - M)$.
- **(b)** What is the birth mass of the whale? (i.e. at $t = 0$)
- **(c)** When the whale is one year old, its mass is 10 tonnes. Show that $k = \ln\left(\frac{6}{5}\right)$.
- **(d)** What is the mass of the whale when it is 10 years old (to the nearest tonne)?
- **(e)** If male southern right whales grow to about 55 tonnes and females grow to about 85 tonnes, determine the gender of this whale, giving reasons for your answer.

Solution

(a) $M = 55 - 54e^{-kt}$

Differentiate with respect to t: $\quad \dfrac{dM}{dt} = 54ke^{-kt}$

Rewrite $M = 55 - 54e^{-kt}$: $\quad 54e^{-kt} = 55 - M$

Substitute into $\dfrac{dM}{dt} = 54ke^{-kt}$: $\quad \dfrac{dM}{dt} = k(55 - M)$

Hence the rate of growth of the whale's mass is $\dfrac{dM}{dt} = k(55 - M)$.

(b) At $t = 0$: $M = 55 - 54e^{0}$

$M = 55 - 54 = 1$

The birth mass of the whale is 1 tonne.

(d) At $t = 10$: $M = 55 - 54e^{-10\ln\left(\frac{6}{5}\right)}$

$M = 55 - 54e^{10\ln\left(\frac{5}{6}\right)}$

$M \approx 46.3$

The mass of the whale to the nearest tonne is 46 tonnes.

(c) At $t = 1$, $M = 10$:

$$\begin{aligned} 10 &= 55 - 54e^{-k} \\ 54e^{-k} &= 45 \\ e^{-k} &= \frac{45}{54} = \frac{5}{6} \\ e^{k} &= \frac{6}{5} \\ k &= \ln\left(\frac{6}{5}\right) \end{aligned}$$

(e) As $t \to \infty$: $M \to 55 - 54e^{-\infty}$

$M \to 55$

The limiting mass of the whale is 55 tonnes, so it is most likely to be a male.

Newton's law of cooling

Newton's law of cooling states that the cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium:

$$\frac{dT}{dt} = -k(T - M)$$

where T is the temperature at any time t
M is the temperature of the surrounding medium (a constant).

Example 24

The original temperature of a body is 100°C, the temperature of its surroundings is 20°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T - 20)$, where T is the temperature of the body at time t, find:

(a) the temperature of the body after 20 minutes

(b) the time taken to cool from 100°C to 60°C.

Solution

$$\frac{dT}{dt} = -k(T - 20)$$

Reciprocal of both sides: $\quad \dfrac{dt}{dT} = \dfrac{-1}{k(T - 20)}, \quad T \neq 20$

Integrate with respect to T: $\quad t = -\dfrac{1}{k}\displaystyle\int \dfrac{1}{T - 20}\,dT$

$$t = -\frac{1}{k}\log_e(T - 20) + C, \quad T > 20$$

$$-k(t - C) = \log_e(T - 20)$$

$$T - 20 = e^{-k(t - C)}$$

Let $A = e^{kC}$ (a constant): $T = 20 + Ae^{-kt}$

When $t = 0$, $T = 100$: $100 = 20 + A$

$A = 80$

When $t = 10$, $T = 70$: $70 = 20 + 80e^{-10k}$

$$e^{-10k} = \frac{50}{80} = 0.625$$

$$-10k = \log_e 0.625$$

$$-10k \approx -0.47$$

$k = 0.047$ (using the approximate logarithm value)

$$\therefore \quad T = 20 + 80e^{-0.047t}$$

(a) When $t = 20$: $T = 20 + 80e^{-0.94}$

$T \approx 51.25$

After 20 minutes the temperature is approximately 51.25°C.

(b) When $T = 60$: $60 = 20 + 80e^{-0.047t}$

$$e^{-0.047t} = \frac{40}{80} = 0.5$$

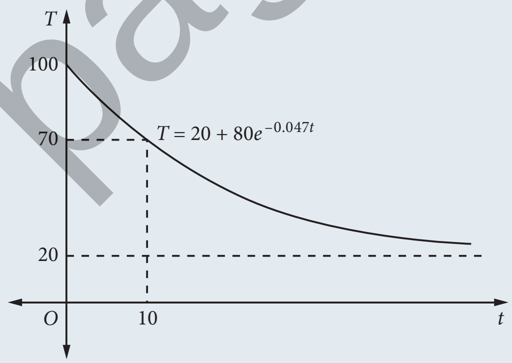
$$-0.047t = \log_e 0.5$$

$$-0.047t \approx -0.6931$$

$$t \approx 14.7$$

The temperature reaches 60°C after approximately 14.7 minutes.

The graph of $T = 20 + 80e^{-0.047t}$ shows that the temperature of the body never falls below the temperature of the surroundings. As $t \to \infty$, $T \to 20$ from above.



Wilhelmy's law

Many chemical reactions follow a law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time:

$$\frac{dx}{dt} = k(a - x), \ 0 \le x \le a$$

where a is the initial concentration and x is the amount transformed at time t.

Example 25

A chemical reaction follows the rule $\frac{dx}{dt} = k(a - x)$, where a is the initial concentration and x is the amount of the reagent transformed at time t. Thus when $t = 0$, $x = 0$. If $a = 10$ and after 2 minutes $x = 4$, find the concentration of the reagent after 5 minutes.

Solution

$$\frac{dx}{dt} = k(10 - x),\ \ 0 \le x \le 10$$

Take the reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{k(10 - x)},\ \ 0 \le x < 10$

Integrate with respect to x: $t = \frac{1}{k}\int \frac{1}{10 - x}\,dx$

$$kt = -\log_e(10 - x) + C$$

When $t = 0$, $x = 0$: $0 = -\log_e 10 + C$

$$C = \log_e 10$$

$$\therefore\ \ kt = \log_e 10 - \log_e(10 - x)$$

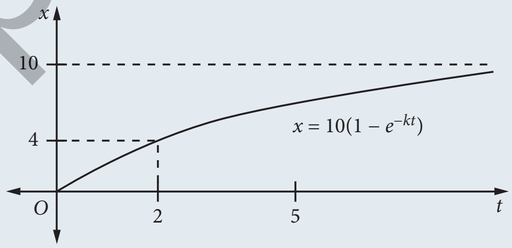
$$kt = \log_e \frac{10}{10 - x}$$

Use inverse functions: $\frac{10}{10 - x} = e^{kt}$

$$\frac{10 - x}{10} = e^{-kt}$$

$$10 - x = 10e^{-kt}$$

$$x = 10(1 - e^{-kt}) \text{ for } t \ge 0$$



Note: The asymptote of the graph is $x = 10$, approached from below. This is consistent with the restriction $0 \le x < 10$ for $t \ge 0$.

When $t = 2$, $x = 4$: $4 = 10(1 - e^{-2k})$

$$0.4 = 1 - e^{-2k}$$

$$e^{-2k} = 0.6$$

Reciprocal of both sides: $e^{2k} = \frac{5}{3}$

$$2k = \log_e\left(\frac{5}{3}\right)$$

$$k = \frac{1}{2}\log_e\left(\frac{5}{3}\right)$$

When $t = 5$: $x = 10(1 - e^{-5k})$

But: $e^{-2k} = 0.6$

$$e^{-5k} = \left(e^{-2k}\right)^{\frac{5}{2}} = \left(\frac{3}{5}\right)^{\frac{5}{2}} \approx 0.279$$

$$\therefore\ \ x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

or inverse functions: $-2k = \log_e 0.6$

$$-2k = -0.51$$

$$k = 0.255$$

i.e. $x = 10(1 - e^{-5k})$

$$x = 10(1 - e^{-1.275})$$

$$x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

$$10 - x = 2.79$$

Hence the concentration is 2.79 units after 5 minutes. (Remember that $(10 - x)$ is the concentration remaining.)

From the previous examples, it can be seen that if $k < 0$, then as $t \to \infty$, $N \to P$.

Given $\frac{dN}{dt} = -k(N - P)$, where k and P are constants, if $k < 0$, then $\lim_{t \to \infty} N = P$.

EXERCISE 14.5 HARDER EXPONENTIAL GROWTH AND DECAY

1 N is decreasing according to the equation $\frac{dN}{dt} = -0.4(N - 30)$. If $N = 60$ when $t = 0$:

(a) show that $N = 30 + Ae^{-0.4t}$ is a solution of this equation, where A is a constant
(b) calculate the value of N when $t = 5$.

2 N is increasing according to the equation $\frac{dN}{dt} = 0.2(N - 40)$. If $N = 50$ when $t = 0$:

(a) show that $N = 40 + Ae^{0.2t}$ is a solution to this equation, where A is a constant
(b) calculate the value of N when $t = 10$.

3 The original temperature of a body is 120°C, the temperature of its surroundings is 50°C and the body cools to 70°C in 10 minutes. Assuming Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T - 50)$ where T is the temperature of the body at time t, find:

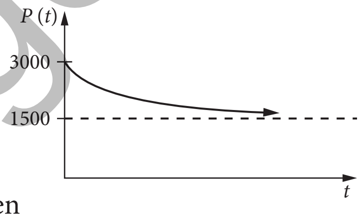
(a) the temperature after 20 minutes **(b)** the time taken to cool to 60°C.

4 If $N = 70$ when $t = 0$, which expression is the correct solution to $\frac{dN}{dt} = -0.5(N - 20)$?

A $N = 20 + 50e^{0.5t}$ **B** $N = 20 + 50e^{-0.5t}$ **C** $N = 20 - 50e^{0.5t}$ **D** $N = 20 - 50e^{-0.5t}$

5 The diagram shows the number of penguins, $P(t)$, on an island at time t.

Which equation represents this graph?



A $P(t) = 1500 + 1500e^{-kt}$ **B** $P(t) = 3000 - 1500e^{-kt}$
C $P(t) = 3000 + 1500e^{-kt}$ **D** $P(t) = 4500 - 1500e^{-kt}$

6 A metal bar has a temperature of 1230°C and cools to 1030°C in 10 minutes when the surrounding temperature is 30°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T - 30)$ where T is the temperature of the body at time t.

(a) Show that $T = 30 + 1200e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.
(b) Find the temperature after 20 minutes.
(c) Find the time taken to cool from 1230°C to 80°C.

7 Water at 20°C is placed in a freezer where the air is at a constant temperature of −10°C. The temperature of the water falls to 15°C in 5 minutes. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T + 10)$, where T is the temperature of the body at time t.

(a) Show that $T = -10 + 30e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.
(b) Find the temperature of the water after another 5 minutes (when $t = 10$).

8 A body whose temperature is 180°C is immersed in a liquid that is at 60°C. In 1 minute the temperature of the body has fallen to 120°C. Assume Newton's law of cooling, i.e. $\frac{dT}{dt} = -k(T - 60)$ where T is the temperature of the body at time t.

(a) Show that $T = 60 + 120e^{-kt}$ satisfies both Newton's law of cooling and the initial conditions.
(b) At what time would the temperature of the body have fallen to 90°C?

9 A current of i amperes (or 'amps') flows through a coil of inductance, L, henrys and resistance, R, ohms. The current at any time is given by $i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$, where E is the electromotive force (i.e. the voltage) in volts. Show that $L\frac{di}{dt} + Ri = E$.

10 A vessel is filled at a variable rate so that the volume of liquid in the vessel at any time, t, is given by $V = A(1 - e^{-kt})$.

(a) Show that $\frac{dV}{dt} = k(A - V)$.
(b) If a quarter of the vessel is filled in the first 5 minutes, what fraction is filled in the next 5 minutes?
(c) Show that $\lim\limits_{t \to \infty} V = A$.

11 A rectangular vessel is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 20 cm, flows into the other compartment, initially empty, at a rate proportional to the difference between the levels in each compartment. The differential equation for this process is $\frac{dx}{dt} = k(20 - 2x)$, where x centimetres is the depth of the liquid in one of the vessels at any time, t minutes.

(a) Show that $x = 10(1 - e^{-2kt})$.

(b) If the level in the second compartment rises 2 cm in the first 5 minutes, at what time will the difference in levels be 2 cm?

12 In a certain chemical process, the amount of a certain substance, measured in y grams, at time, t, hours is given by the formula $y = 3 + e^{-kt}$.

(a) Show that $\frac{dy}{dt} = -k(y - 3)$.

(b) If initially y decreases at a rate of 0.08 grams per hour, find the value of k.

(c) Find the rate of change when $y = 3.5$.

(d) What values can y take?

13 On a 36°C day, Xiao takes an ice cream out of the freezer. The freezer is set at an optimal temperature of −18°C. After one minute, the temperature of the ice cream is −10°C.

The differential equation $\frac{dT}{dt} = -k(T - 36)$ models this situation, where T is the temperature of the ice cream and t is the time in minutes out of the freezer.

(a) Show that the equation $T = 36 + Ae^{-kt}$, where A is a constant, satisfies the differential equation provided.

(b) It is known that ice cream begins to melt at 0°C. After how many minutes will the ice cream begin to melt? Give your answer to one decimal place.

14 The population of dingoes on an island is modelled by the logistic equation $\frac{dy}{dt} = y(1 - y)$, where y is the fraction of the island's carrying capacity of dingoes reached after t years.

Initially, the population of dingoes is estimated to be one-quarter of the island's carrying capacity.

(a) Use the substitution $y = \frac{1}{1 - w}$ to transform the logistic equation to $\frac{dw}{dt} = -w$.

(b) **(i)** Show that $w = Ae^{-t}$ is a solution to $\frac{dw}{dt} = -w$.

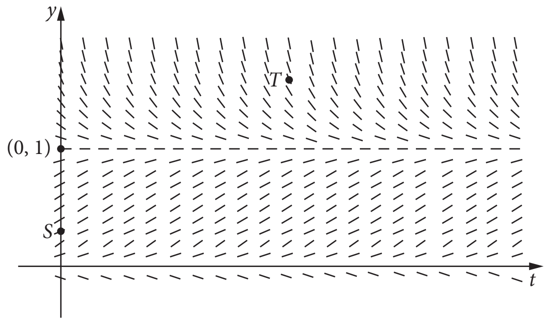
(ii) Using the solution of $\frac{dw}{dt} = -w$, find the solution of the logistic equation for y satisfying the initial conditions.

(c) Show that when $t = \ln 9$, the dingo population is three-quarters of the island's carrying capacity.

15 The population of dingoes on the island in question **14** is once again modelled by the logistic equation $\frac{dy}{dt} = y(1 - y)$, where y is the fraction of the island's carrying capacity of dingoes reached after t years.

Initially, the population of dingoes is estimated to be 30% of the island's carrying capacity.

The diagram shows a direction field for this differential equation, with S representing the initial population.



(a) Explain why the graph of the solution that passes through the point S cannot also pass through the point T.

(b) On a copy of the diagram, clearly sketch the graph of the solution that passes through the point S.

(c) Find the predicted value of the fraction of the population, y, at which the rate of growth of the fraction of the population, y, is greatest.

16 A hard-boiled egg at 98°C is put in a room at 18°C. After 5 minutes, the egg's temperature is 58°C. The rate of cooling of the egg is proportional to the difference between the temperature, T, of the egg and the temperature, S, of the room, i.e. $\frac{dT}{dt} = k(T - S)$, where k is a constant and t is time in minutes.

(a) Show that $k = -\frac{\ln 2}{5}$.

(b) When will the temperature of the egg drop to 20°C? Give your answer to the nearest minute.

17 In ceramic kilns, the thermocouple is the device that reports the temperature to the user.

In an old kiln, the thermocouple is faulty and fails somewhere around the 900°C mark, meaning the user is unable to determine whether the kiln is correctly reaching higher temperatures.

Six hours after the kiln begins to cool, the temperature is recorded at 750°C. A second measurement of 500°C was recorded after another three hours.

The rate of change of the temperature in the kiln is proportional to the difference between its temperature and the room temperature of 80°F. This can be modelled using Newton's law of cooling, $\frac{dT}{dt} = -k(T - A)$, where T is the temperature in degrees at time t hours after the kiln has begun to cool.

Calculate the initial temperature of the kiln.

18 A team of biologists released 500 fish into a lake with a maximum carrying capacity of 10 000 fish. The fish population, N, in the lake after t years is modelled by the logistic equation $\frac{dN}{dt} = kN(10\,000 - N)$, where k is a constant.

(a) Given $\frac{10\,000}{N(10\,000 - N)} = \frac{1}{N} + \frac{1}{(10\,000 - N)}$, solve the differential equation to show that the fish population at t years is $N = \frac{10\,000}{1 + 19e^{-10\,000kt}}$.

(b) It was found that the number of fish tripled during the first year, hence $k = \frac{1}{10\,000}\ln\left(\frac{57}{17}\right)$.

Find, correct to the nearest month, how long it will take for the fish population in the lake to reach 7000 fish.

19 Water is being heated in a kettle. At time t seconds, the temperature of the water is T°C.

The rate of increase of the temperature of the water at any time after the kettle is switched on is modelled by the equation $\frac{dT}{dt} = k(120 - T)$, where k is a positive constant.

The temperature of the water is 20°C when the kettle is switched on.

(a) Show that $T = 120 - 100e^{-kt}$ is both a solution to the differential equation and satisfies the initial condition.

(b) When the temperature of the water reaches 100°C, the kettle switches off.
If it takes 10 seconds for the temperature to reach 30°C, once the kettle is switched on, find how long it takes for the kettle to switch off, to the nearest second.

20 The rate of change of a tadpole population is given by the differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$, where P is the number of tadpoles after t days and r and k are positive constants. Initially the pond is home to P_0 tadpoles.

(a) Show that the tadpole population is increasing when $0 < P < k$.

(b) Solve the differential equation to find P as a function of t.

You may assume that $\frac{k}{P(k - P)} = \frac{1}{P} + \frac{1}{k - P}$.

(c) Find $\lim_{t \to \infty} P$, which is the maximum number of tadpoles that the pond can support.

(d) Find an expression for the time when the rate of change of population is greatest.

21 On a certain island, a researcher released 400 mosquitoes. The population growth is given by the logistic equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{16\,000}\right)$, where P is the population of mosquitoes, t is the number of years after the mosquitoes were released, and r is a constant.
Five years after the mosquitoes were released, the population of mosquitoes was estimated to be 4600.

It is given that $\frac{1}{P(16\,000 - P)} = \frac{1}{16\,000}\left(\frac{1}{P} + \frac{1}{16\,000 - P}\right)$.

(a) Show by integration that $P = \frac{16\,000}{39e^{-rt} + 1}$.

(b) Hence show that $r = -\frac{1}{5}\ln\left(\frac{19}{299}\right)$.

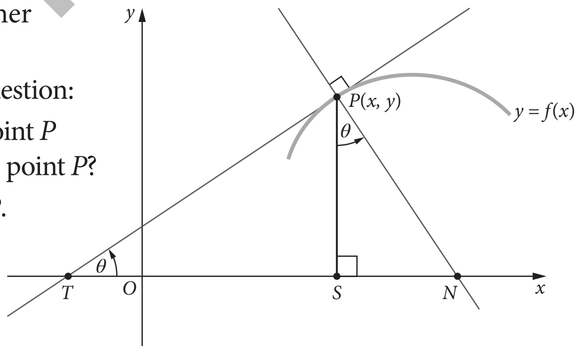
(c) When is the population of mosquitoes growing at its fastest rate?

(d) Find the island's maximum possible population of mosquitoes.

The researcher determines that if the population of mosquitoes exceeds 13 000, it will harm the ecosystem of the island. Spiders are introduced to the island to control the mosquito population, reducing it by 9% every year. The modified logistic equation for the population growth is $\frac{dP}{dt} = rP\left(1 - \frac{P}{16\,000}\right) - 0.09P$.

(e) Determine if the researcher will be able to protect the ecosystem from harm.
Justify your answer with the appropriate calculations.

14.6 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)g(y)$ USING SEPARATION OF VARIABLES



Towards the end of his life, the great Swiss mathematician and teacher Johann Bernoulli (1667–1748) published his lectures on calculus.

In Bernoulli's discussion of integral calculus, he posed the following question: What curve $y = f(x)$ has a subtangent ST (see diagram at right) at any point P that is always equal to a positive multiple, $a > 0$, of the abscissa OS at the point P?

The term abscissa is the formal name for the x-coordinate of point P.

In other words, Bernoulli is asking for the curves with $ST = aOS$.

Bernoulli's question can be rewritten using a more modern notation.

It follows from the right-angled ΔSPT that $ST = \frac{SP}{\tan\theta}$.

However, TP is a tangent line to the curve $y = f(x)$ at point P, given by $\tan\theta = \frac{dy}{dx}$.

SP is the ordinate (the y-value) of point P.

Therefore: $ST = \frac{PS}{\tan\theta} = \frac{y}{\left(\frac{dy}{dx}\right)}$

$ST = aOS \therefore \frac{y}{\frac{dy}{dx}} = ax$

Solving for the derivative in this equation gives $\frac{dy}{dx} = \frac{y}{ax}$.

In some ways, this problem is slightly more complicated than the equations solved so far because it has the form $\frac{dy}{dx} = f(x)g(y)$. Both the independent and the dependent variables appear as separate factors in the derivative $\frac{dy}{dx}$.

In this section you will learn how to answer Bernoulli's question. You will learn how to solve any first-order differential equation of the form $\frac{dy}{dx} = f(x)g(y)$. The key to solving these equations is the separation of the two variables onto either side of the equality.

In other words, to solve $\frac{dy}{dx} = f(x)g(y)$, transpose the equation so that all terms involving the dependent variable are on the left side of the equality and all terms involving the independent variable are on the right side.

This would give $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$.

In the table below, the original equation is in the left-hand column while the equation with variables separated is in the right-hand column.

Original differential equation	Variables	Rewritten with variables separated
$\cos x + y\frac{dy}{dx} = 0$	x and y	$y\frac{dy}{dx} = -\cos x$
$\sin v\frac{dw}{dv} - \cos v = 0$	v and w	$\frac{dw}{dv} = \cot v$
$\theta z' = z + 1$	θ and z	$\frac{1}{z+1}\frac{dz}{d\theta} = \frac{1}{\theta}$

Finding the general solution of a first-order differential equation by the method of separation of variables

Example 26

Find the general solution of the differential equation $\frac{dy}{dx} = x(y-1)$.

Solution

The dependent and independent variables are separated onto either side of the equality, with the dependent on the left and the independent variable on the right: $\frac{1}{y-1}\frac{dy}{dx} = x$.

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx = dy$ on the LHS:

$$\int \frac{1}{y-1}\frac{dy}{dx}\,dx = \int x\,dx$$

$$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$$\log_e|y-1| = \frac{1}{2}x^2 + c$$

Represent the dependent variable as an explicit function of the independent variable.

Exponentiating both sides:

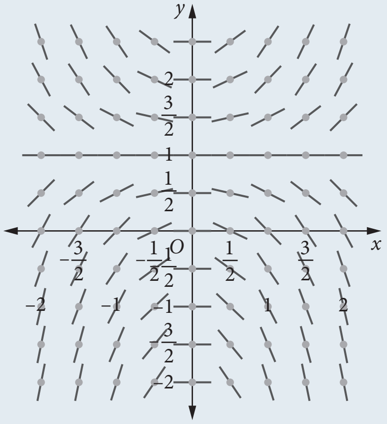
$$|y-1| = e^{\frac{1}{2}x^2 + c}$$

$$= e^c e^{\frac{1}{2}x^2}$$

Removing the absolute value: $y = 1 + Ae^{\frac{1}{2}x^2}$, where $A = \pm e^c$.

It appears that the general solution of $\frac{dy}{dx} = x(y-1)$ is $y = 1 + Ae^{\frac{1}{2}x^2}$, Real A, $A \neq 0$.

The slope field of $\frac{dy}{dx} = x(y-1)$ is shown at right.



This slope field indicates that $y=1$ should also be a solution curve for $\frac{dy}{dx}=x(y-1)$. However, the general solution $y=1+Ae^{\frac{1}{2}x^2}$ cannot give $y=1$ because $A=\pm e^c$, $A\neq 0$.

In the example above, the solution $y=1$ is an example of a so-called singular solution because it is not part of the general solution for any allowable value of the constant of integration. However, the singular solutions of $\frac{dy}{dx}=f(x)g(y)$ are often of the form $y=y^*$ for some root k of the equation $f(y^*)=0$. The singular solutions are also usually evident from an investigation of the slope field.

Warning

To avoid missing any possible singular solutions when finding the general solution of a differential equation of the form $\frac{dy}{dx}=f(y)g(x)$, always remember to investigate the roots $\{y^*: f(y^*)=0\}$ and/or the slope field.

Finding the particular solution of a first-order differential equation by the method of separation of variables

Example 27

Find the particular solution of Bernoulli's problem $\frac{dy}{dx}=\frac{2y}{x}$ passing through the point $(1, -4)$.

Solution

The dependent and independent variables are separated onto either side of the equality with the dependent on the left and the independent variable on the right: $\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}$

Both sides of the equality are integrated with respect to the independent variable, using the change of variable $\frac{dy}{dx}dx=dy$ on the LHS of the equation: $\int\frac{1}{y}\frac{dy}{dx}dx=2\int\frac{1}{x}dx$

$$\int\frac{1}{y}dy=2\int\frac{1}{x}dx$$

$$\log_e|y|=2\log_e x+c$$

$$\log_e|y|=\log_e x^2+c$$

Solved for the dependent variable to obtain the general solution: $\log_e|y|-\log_e x^2=c$

$$\log_e\frac{|y|}{x^2}=c$$

$$\frac{|y|}{x^2}=e^c$$

$\therefore\ y=Ax^2$, where $A=\pm e^c$

Constant of integration is found to satisfy the required initial condition $(x, y)=(1, -4)$: $-4=A$

Constant of integration is substituted to specify the particular solution of the problem: $y=-4x^2$.

To solve the first-order differential equation $\frac{dy}{dx} = f(y)g(x)$ by the method of separation of variables:

1 Solve $\{ y^* : f(y^*) = 0\}$ for any steady state (or equilibrium) solutions $y = y^*$.

2 Separate the dependent and independent variables onto either side of the equality $\frac{1}{f(y)}\frac{dy}{dx} = g(x)$.

3 Integrate both sides of the equality with respect to the independent variable:

For the integral involving y terms, apply the change of variable $\int \frac{1}{f(y)}\frac{dy}{dx}\,dx = \int \frac{1}{f(y)}\,dy$.

Add the constant of integration C to the side with the independent variable. This is the general solution.

4 Wherever possible, you should represent the dependant variable as an explicit function of the independent variable. However, you may need to be satisfied with an equation that determines the dependant variable as an implicit function of the independent variable.

5 If an initial condition $y(a) = b$ is given, solve for the constant of integration C. This is the particular solution.

You have now learnt two different methods to solve the differential equation $\frac{dy}{dx} = f(y)$.

The reciprocal method

1 Take the reciprocal of both sides of $\frac{dy}{dx} = f(y)$ to obtain $\frac{dx}{dy} = \frac{1}{f(y)}$.

2 Integrate the result with respect to y to obtain $\int \frac{dx}{dy}\,dy = \int \frac{1}{f(y)}\,dy$.

The separation of variables method

1 Separate the two variables in $\frac{dy}{dx} = f(y)$ to obtain $\frac{1}{f(y)}\frac{dy}{dx} = 1$.

2 Integrate the result with respect to x to obtain $\int \frac{1}{f(y)}\frac{dy}{dx}\,dx = \int dx$ or $\int \frac{1}{f(y)}\,dy = \int dx$.

Each of the two methods above give the original independent variable x as a function of the original dependent variable y, which must then be inverted to give the required solution $y(x)$.

EXERCISE 14.6 SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)g(y)$ USING SEPARATION OF VARIABLES

1 Use the method of separation of variables to find the general solution of each of the differential equations below. Where reasonable, express the family of solutions as explicit functions of x.

(a) $\left(x^2+4\right)\frac{dy}{dx} = 2xy$

(b) $\frac{dy}{dx} = \frac{2y}{x}$

(c) $\frac{dy}{dx} = \left(1+y^2\right)\sqrt{x}$

(d) $\frac{\sqrt{1-x^2}}{y+1}\frac{dy}{dx} = -2x$

(e) $\left(1+x^2\right)\frac{dy}{dx} = xy$

(f) $e^y \cos x - \frac{dy}{dx}\sin^2 x = 0$

(g) $(\sec x)\, y' + y^2 = 0$

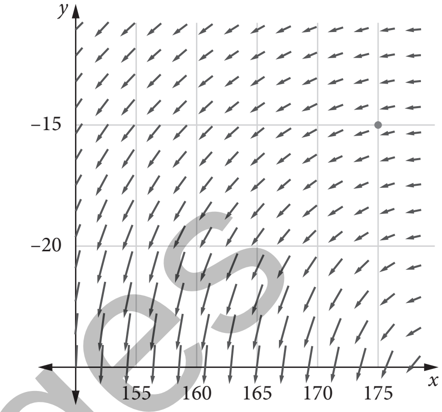
2 Find the particular solution of $e^{-x^2} yy' + xy = 0,\ y(0) = 1$.

3 Find the equation of each graph:

(a) The graph passes through (1, 2) and has a slope $\frac{3y}{x^2}$ at each point (x, y).

(b) The gradient of the tangent at point (x, y) on a graph is given by $\frac{-2y}{x}$ and the graph passes through the point (1, 2).

4 On a particular day, the synoptic scale airflow over the Coral Sea is modelled by the slope field of the following differential equation $\frac{dy}{dx} = \frac{180-x}{4(25+y)}$, as shown in the diagram at right.



Here, x is the longitude and y is the latitude (both are measured in degrees, with southern latitudes negative).

Tropical Cyclone Wendy is shown at the point longitude 175°E and latitude 15°S.

(a) Find the particular solution of $\frac{dy}{dx} = \frac{180-x}{4(25+y)}$, $y(175) = -15$.

(b) Hence, plot the path of Cyclone Wendy.

5 Consider the differential equation $\frac{dy}{dx} = 3x^2 \cos^2 y$.

(a) Find the particular solution $y = f(x)$ to the differential equation, satisfying the initial condition $f(0) = \frac{\pi}{4}$.
(b) State the domain and range of the solution found in part **(a)**.

6 An insect population P experiences a seasonal growth rate given by $\frac{dP}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right)P$, $P(0) = 1$, where P is measured in millions and t is the number of months since the beginning of spring.

Express the time variation of the insect population P and sketch this variation over the course of one year.

7 The general solution of the differential equation $\frac{dy}{dx} = y^2 \cos x$ is:

A $y = \frac{1}{\cos x - C}$
B $y = \frac{-1}{\sin x + C}$
C $y = \sqrt{2C + 2\sin x}$
D $y = -\sqrt{2C + 2\sin x}$

8 The particular solution of the differential equation $\frac{dy}{dx} = 2xy$, $y(0) = 2$ is:

A $y = e^{\frac{x^2}{2}}$
B $y = 2e^{x^2}$
C $y = c_1 e^{x^2}$
D $y = 2e^{\frac{x^2}{2}}$

9 Consider the differential equation $\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{1-\sin x}$, where $0 \le x < \frac{\pi}{2}$ and $y > 0$.

Given that $y = 1$ when $x = \frac{\pi}{6}$, express y as a function of x.

10 Consider the differential equation $\frac{dy}{dx} = xy - 2x$.

(a) Find the gradient of the tangent to the solution curve that passes through the point (2, 3).
(b) Find the equation of the solution curve to the DE that passes through the point (2, 3).

11 Find the solution to the differential equation $\frac{dy}{dx} = 2xy$, given that the graph passes through the point (0, 2).

Challenging

12 The altimetry equation $\frac{dp}{dz} = -\frac{g}{R}\frac{P}{(T_0 - \Gamma z)}$, relates air pressure P in pascals (Pa) to the altitude (height above sea level) z in metres (m). In this model, T_0 is the air temperature in kelvin (K) at sea level; Γ is the constant rate of temperature change as the altitude increases (K m^{-1}); P_0 is the air pressure (Pa) at sea level; R is the gas constant of air; and g is the gravitational acceleration.

(a) Find the rule for the air pressure, P, as a function of the altitude, z.
(b) Hence, find the rule for altitude, z, as a function of the air pressure, P.
(c) Use the table below to find the altitude corresponding to an air pressure of 61 640 Pa. Express the answer correct to the nearest metre.

g (m s^{-2})	R (J kg^{-1}K^{-1})	Γ (K m^{-1})	P_0 (Pa)	T_0 (K)
9.80	287.053	0.0065	101 325	288.15

13 Consider the differential equation $\frac{dy}{dx} = -\frac{2xy}{\log_e y}$, $y > 0$.

(a) Find the general solution $g(x, y) = c$ of this differential equation as an implicit relation between x and y, using the substitution $u = \log_e y$ to complete the integration.
(b) Find the particular solution passing through the point $(0, e)$.
(c) Explain why $x = 1$ cannot exist in the solution to part **(b)**.

14 Consider the differential equation: $\frac{dy}{dx} = \frac{5-x}{y}$

(a) Let $y = f(x)$ be the particular solution to the differential equation $0 < x < 10$, such that the line $y = 5$ is a tangent to the graph of $y = f(x)$. Find the x-coordinate of the point P of tangency.
(b) Given that $\frac{d^2y}{dx^2} = -\frac{1}{y} - \frac{(5-x)^2}{y^3}$, determine the nature of the solution of the original differential equation in the vicinity of point P.
(c) Let $y = g(x)$ be the particular solution to the given differential equation for $0 < x < 10$, with the initial condition $g(5) = -5$. Find the rule $y = g(x)$.

15 The fuel–air pressure, P (Pa), inside the cylinder of a combustion engine with a 10 : 1 compression ratio is related to the volume, V (cm^3), of the cylinder by the differential equation $\frac{dP}{dV} = -1.400\frac{P}{V}$.

(a) Find the particular solution of this equation, if the volume of the fuel–air in the uncompressed cylinder is 600 cm^3 when the fuel–air pressure is 100 000 Pa.
(b) Hence, find the air pressure, P, when the cylinder is fully compressed to 60 cm^3.

16 Tropical cyclones can only form over water with surface temperatures of at least 26°C. After these cyclones have formed they tend to strengthen when passing over warmer water and weaken over colder water.

The strength of a cyclone S is related to the water surface temperature T by $\frac{dS}{dT} = k\frac{S}{T}$ where k is the constant of proportionality.

(a) What is the sign of the constant of proportionality k? Explain why this is so.
(b) Find $S(T)$ in terms of k.

17 A space probe is launched vertically upwards from the surface of a spherical planet with a radius R. If the atmospheric drag is ignored, the upwards velocity, v, (m s^{-1}) of the probe at height h metres above the surface of the planet is modelled by the solution of the differential equation $\frac{dv}{dh} = -\frac{gR^2}{v(R+h)^2}$, $v = u$, where $h = 0$ and g is the gravitational acceleration on the surface of the planet.

(a) Show that $v^2 = u^2 - \frac{2gR}{\left(1 + \frac{R}{h}\right)}$.
(b) Hence find the minimum launch velocity, u, for the probe to escape the planet's gravity.

18 In a dry, still atmosphere, the air temperature T(K) usually decreases with the altitude h (m) above sea level according to $\frac{dT}{dh} = -0.006\,49$, with $T(0) = 288$ K.

(a) Find a formula for the variation of temperature with height.

The barometric pressure P (Pa) also depends on the altitude above sea level h and the local air temperature T according to the differential equation $\frac{dP}{dh} = -0.0352\frac{P}{T}$, $P(0) = 101\,325$ Pa.

Whereas the temperature, B, at which water boils is related to air pressure, P, by the differential equation $\frac{dB}{dP} = 0.000189\frac{B^2}{P}$, with $B(101\,325) = 373$ K.

(b) Show that $\frac{dB}{dh} = \frac{-6.65\times 10^{-6}B^2}{288 - 0.006\,49h}$.

(c) Hence find a formula for the altitude, h (m), as a function of the boiling point temperature B (K).

During a climb of Mount Everest, mountaineers observe water boiling at 84°C (that is, 357 K).

(d) Find the height of the mountaineers above sea level.

19 According to Torricelli's Law, the rate of outflow from a tank filled to a depth, h, with a volume V, of fluid draining under the influence of gravity alone is $\frac{dV}{dt} = -k\sqrt{h}$, where $k > 0$.

Consider a hemispherical tank of radius 1 m, filled to a depth h m with $V(h) = \frac{1}{3}\pi h^2(3 - h)$ cubic metres of water. Assume that this tank drains through a hole at its lowest point.

(a) Use the method of related rates to show that: $\frac{dh}{dt} = -\frac{k}{\pi\sqrt{h}(2-h)}$.

(b) Initially, the tank is filled to a depth of 1 m. Find a formula for the time t taken for the water level to fall to a given depth, h. (Express your answer in terms of the unknown constant of proportionality, k.)

(c) Find the constant of proportionality k, if it takes $\frac{187}{16}$ minutes for the depth to fall to $\frac{1}{4}$ m.

(d) Hence find the time taken (in minutes) for the tank to empty.

14.7 MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

Models of uninhibited growth

The British economist Thomas R. Malthus (1766–1834) proposed a model of natural population growth that assumes that both the birth rate and the death rate of a population are proportional to the current size of the population at each instant.

Consequently, the net rate of change in population can be restated as:
'net rate of change in population = birth rate – death rate'.

In other words: Let $P(t)$ represent the size of a population at time t. Assuming that the population level is specified at the initial time $t = 0$ as $P(0) = P_0$, predict the population at later times t.

$$\frac{dP}{dt} = bP - dP$$

$$\frac{dP}{dt} = (b - d)P$$

$$\frac{dP}{dt} = rP, \text{ where } r = b - d \text{ with } P(0) = P_0.$$

In summary, Malthus' uninhibited growth model $\frac{1}{P}\frac{dP}{dt} = r$ has a constant relative growth rate r. This model results in exponential growth for P. In more general terms:

> A dependent variable $y(t)$ is said to have an *exponential growth rate* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = k > 0$, or an *exponential rate of decay* if its relative growth rate $\frac{1}{y}\frac{dy}{dt} = -k < 0$ for some positive constant, k.

Finding the doubling-time in a model of exponential growth

Example 28

Consider the differential equation $\frac{dy}{dt} = ky$, given that $y(0) = y_0 > 0$, with $k > 0$.

Find the time T for y to double its initial value y_0.

Solution

Rearrange to separate the two variables:

$$\frac{dy}{dt} = k\,y$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

Integrate with respect to the independent variable:

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int k dt$$

$$\log_e |y| = kt + c$$

Exponentiate: $|y| = e^{kt+c}$

The general solution is: $y = Ae^{k\,t}$ where $A = \pm e^c$

Apply the initial condition $y(0) = y_0$ to solve for the constant of integration: $y_0 = A$

Substitute to find the model of exponential growth: $y = y_0 e^{k\,t}$

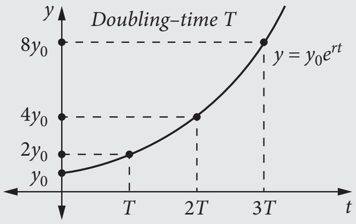
At time T, $y = 2y_0$:

$$2y_0 = y_0 e^{k\,T}$$

$$2 = e^{kT}$$

$$T = \frac{1}{k}\log_e 2$$

For a system $y = y_0\,e^{r\,t}$, $r > 0$ experiencing exponential growth, the doubling-time (T) is the time it takes the system to double in size. The formula for the doubling-time is $T = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2$



Radioactive decay

Example 29

The New Zealand-born physicist Ernest Rutherford is often called the father of nuclear physics. Rutherford was able to formulate a model of radioactive decay as a simple differential equation. From his laboratory experiments Rutherford discovered that the relative rate r, at which atoms of radioactive materials disintegrate, is proportional to the number of atoms N still present in the sample. In other words, this model states that $\frac{dN}{dt} = -rN$, $N(0) = N_0$.

(a) Show that the solution of this model is $N = N_0 e^{-r\,t}$.

(b) Hence find the time $t_{\frac{1}{2}}$ required for the number of atoms still present (N) to fall to $\frac{1}{2}N_0$.

Solution

(a) The two variables are separated: $\frac{1}{N}\frac{dN}{dt} = -r$

Both sides of the equation are integrated with respect to the independent variable:

$$\int \frac{1}{N}\frac{dN}{dt}\,dt = -\int r\,dt$$

$$\log_e |N| = -rt + c$$

$$N = \pm e^{-rt+c}$$

$$N = \pm e^{c} e^{-rt}$$

$$\therefore \quad N = Ae^{-rt} \quad \text{where } A = \pm e^{c}$$

The initial conditions are substituted in the last equation to find the constant of integration, $N(0) = N_0$, $N_0 > 0$:

$A = \pm e^{c} = N_0$

$\therefore N = N_0 e^{-rt}$ as required.

(b) The conditions required are substituted into the equation from part **(a)**: $N = N_0 e^{-rt}$ with $N = \frac{1}{2}N_0$

$$\frac{1}{2}N_0 = N_0 e^{-rt}$$

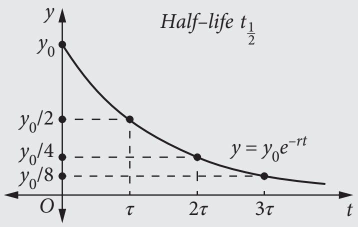
$$\frac{1}{2} = e^{-rt}$$

The exponential equations are changed into a logarithmic form: $t = \frac{1}{r}\log_e 2$.

Hence $t_{\frac{1}{2}} = \frac{1}{r}\log_e 2$, where $t_{\frac{1}{2}}$ is the half-life of the radioactive material.

For a system $y = y_0 e^{-rt}$, $r > 0$ experiencing exponential decay, the half-life, $t_{\frac{1}{2}}$ is the time it takes the value of y to halve. The formula for the half-life is

$$t_{\frac{1}{2}} = \frac{1}{r}\log_e 2 = \frac{(t_2 - t_1)}{\log_e\left(\frac{y(t_1)}{y(t_2)}\right)}\log_e 2.$$



First-order decay reactions

In a simple first-order decay reaction, the rate of decrease $-\frac{d[A]}{dt}$ of the concentration of a reactant A varies directly with the value of the concentration $[A]$ of that reactant.

That is, the concentration $[A]$ of the reactant A is modelled by the differential equation $\frac{d[A]}{dt} = -r[A]$, where the relative decay rate, r, is usually called the reaction rate.

Examples of some first-order decay reactions are included in the table below.

Reactant	Reaction rate r (s^{-1})	Half-life $t_{\frac{1}{2}}$
$^{238}_{92}U$	4.87×10^{-18}	4.51×10^{9} years
$^{14}_{6}C$	3.83×10^{-12}	5.73×10^{3} years
$^{32}_{15}P$	5.61×10^{-7}	14.3 days

Newton's law of cooling

Newton's law of cooling states that the time rate of decrease in the temperature T of an object varies directly with the difference between the object's temperature and the ambient temperature (the temperature of the surrounding medium, T_a).

Therefore: $\frac{dT}{dt} = -r(T - T_a), \quad r > 0, \quad T(0) = T_0$

If $T_0 > T_a$, the body's initial temperature is higher than the temperature of its surrounding, so the body is cooling. The model predicts that the body will cool quickly at first, reducing the excess of its temperature to that of the surrounding environment. This cooling reduces the excess temperature of the body, thus continually slowing its rate of cooling.

However, if $T_0 < T_a$, then the body's initial temperature is lower than the temperature of its surroundings, so the body is warming. Again, this change in temperature reduces the temperature difference with the surrounding medium, thus slowing the rate of warming.

Example 30

Newton's law of cooling can be used to model the temperature of a cup of coffee cooling on a kitchen bench. If the temperature of the kitchen is a constant 20°C and the initial temperature of the coffee is 95°C, after 20 minutes the coffee will have cooled to 65°C.

(a) Find the temperature of the coffee after an additional 20 minutes.

(b) Sketch a graph of the temperature of the coffee for the first 2 hours.

Solution

(a) This problem models the process of cooling, so Newton's law of cooling is: $\frac{dT}{dt} = -r(T - 20)$, $r > 0$, $T(0) = 95$.

The two variables are separated: $\frac{1}{(T-20)}\frac{dT}{dt} = -r$

Both sides of the equation are integrated with respect to t and a change of variable is applied on the LHS of the equation: $\int \frac{1}{(T-20)}\frac{dT}{dt}\,dt = -\int r\,dt$

$$\int \frac{1}{(T-20)}\,dT = -\int r\,dt$$

$$\log_e |T-20| = -rt + c$$

Both sides of the equation are exponentiated and solved for the dependent variable: $T - 20 = Ae^{-rt}$, where $A = \pm e^c$.

The initial condition is substituted and the constant of integration is determined:

$T(0) = 95$: $95 - 20 = A$

$\therefore\ T = 20 + 75e^{-rt}$

$t = 20$, $T = 65$, is substituted to determine the constant of proportionality: $\log_e\left(\frac{45}{75}\right) = -20r$

$$r = \frac{1}{20}\log_e\left(\frac{5}{3}\right)$$

The constant of proportionality is substituted in the general solution: $T = 20 + 75e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)}$

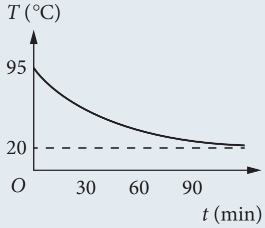
Now: $e^{-\frac{t}{20}\log_e\left(\frac{5}{3}\right)} = e^{\frac{t}{20}\log_e\left(\frac{3}{5}\right)} = e^{\log_e\left(\frac{3}{5}\right)^{\frac{t}{20}}} = \left(\frac{3}{5}\right)^{\frac{t}{20}}$

Simplify: $T(t) = 20 + 75\left(\frac{3}{5}\right)^{\frac{t}{20}}$

After an additional 20 minutes, $t = 40$: $T(40) = 20 + 75\left(\frac{3}{5}\right)^2 = 20 + 27 = 47°\text{C}$

The temperature of the coffee is 47°C.

(b) Using the model from part **(a)** to plot the changing value of T:

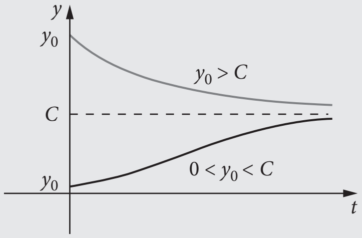


Models of modified growth and decay

The uninhibited growth model with its constant relative growth rate $r = \frac{1}{y}\frac{dy}{dt}$ does not take into account the inherent limitations on the growth of a population. In practice, most populations have a size (the carrying capacity C) beyond which their environment can no longer sustain them. To account for the finite carrying capacity inherent in most systems, the Belgian mathematician Pierre François Verhulst (1804–1849) modified the uninhibited growth model by replacing constant relative growth rate r with a damped growth rate $r\left(1 - \frac{y}{C}\right)$.

Verhulst's modified growth rate model is then equivalent to $\frac{1}{y}\frac{dy}{dt} = r\left(1 - \frac{y}{C}\right)$, which slows the rate of growth as the population approaches the 'carrying capacity' $y = C$ of the system.

The differential equation $\frac{dy}{dt} = ry\left(1 - \frac{y}{C}\right)$, $y(0) = y_0$, where $r > 0$ is called a **logistic equation**. This models population growth in an environment with a finite carrying capacity $C > 0$.



Example 31

Rabbit Island currently has 5000 rabbits, but has sufficient space and food for 20 000 rabbits in total.

The model describing this rabbit population P is then $\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{20\,000}\right)$.

Find the particular solution of this differential equation, given that $\frac{20\,000}{P(20\,000 - P)} = \frac{1}{P} + \frac{1}{20\,000 - P}$.

Solution

Take the reciprocal of both sides of the differential equation: $\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{20\,000}\right)$

$$\frac{dt}{dP} = \frac{200\,000}{P(20\,000 - P)}$$

Integrate both sides of the equation with respect to the original dependent variable:

Given that $\dfrac{200\,000}{P(20\,000-P)} = 10\left(\dfrac{1}{P}+\dfrac{1}{20\,000-P}\right)$

$$\int \frac{dt}{dP}\,dP = \int \frac{200\,000}{P(20\,000-P)}\,dP$$

$$\frac{1}{10}\int dt = \int\left(\frac{1}{P}+\frac{1}{20\,000-P}\right)dP$$

$$\frac{t}{10}+c = \log_e\left|\frac{P}{20\,000-P}\right|$$

Find the general solution: $Ae^{\frac{t}{10}} = \dfrac{P}{20\,000-P}$, where $A = \pm e^{c}$.

Substitute the initial conditions $t = 0$, $P = 5000$: $A = \dfrac{5000}{20\,000-5000} = \dfrac{1}{3}$

$$P = \frac{20\,000e^{\frac{t}{10}}}{3+e^{\frac{t}{10}}}$$

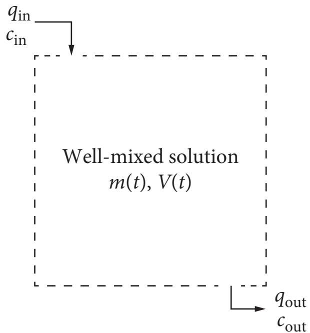
$$= \frac{20\,000}{1+3e^{\frac{-t}{10}}}$$

Modelling the time dependence of a conserved substance

Differential equations can be used to model the time behaviour of a conserved quantity within a confined region of space, if you know the rate at which the quantity flows into and out of the region. Such models are of great practical value in the fields of Science and Engineering.

If a substance m is conserved (that is, neither created nor destroyed), then the rate of change of m within a confined region (called a *control volume*) is equal to the difference between the rate at which it flows into the region (R_{in}), and the rate at which it flows out of the region (R_{out}). That is, $\dfrac{dm}{dt} = R_{\text{in}} - R_{\text{out}}$.

Consider the situation shown at right:



- $m(t)$ is the mass of some conserved substance (e.g. salt or sugar) in the control volume at time, t
- $V(t)$ is the volume of the well-mixed solution within the control volume at time t
- q_{in}, measured in units of $\left(\dfrac{\text{volume}}{\text{time}}\right)$, is the rate at which the solution flows into the control volume
- q_{out}, measured in units of $\left(\dfrac{\text{volume}}{\text{time}}\right)$, is the rate at which the solution flows out of the control volume
- c_{in}, measured in units of $\left(\dfrac{\text{mass}}{\text{volume}}\right)$, is the concentration of the conserved substance in the inflow
- c_{out}, measured in units of $\left(\dfrac{\text{mass}}{\text{volume}}\right)$, is the concentration of the conserved substance in the outflow

The rate R_{in} at which the conserved substance $m(t)$ flows into the control volume is measured in units of $\left(\dfrac{\text{mass}}{\text{time}}\right)$ so that $R_{\text{in}}\left(\dfrac{\text{mass}}{\text{time}}\right) = q_{\text{in}}\left(\dfrac{\text{volume}}{\text{time}}\right)\times c_{\text{in}}\left(\dfrac{\text{mass}}{\text{volume}}\right)$

Similarly, the rate at which $m(t)$ flows out of the control volume is $R_{\text{out}}\left(\dfrac{\text{mass}}{\text{time}}\right) = q_{\text{out}}\left(\dfrac{\text{volume}}{\text{time}}\right)\times c_{\text{out}}\left(\dfrac{\text{mass}}{\text{volume}}\right)$

Assume that $m(t)$ is conserved, so it is neither created nor destroyed within the control volume. Therefore $\frac{dm}{dt}=R_{\text{in}}-R_{\text{out}}$.

This means that the net rate of change in the mass of the solute $m(t)$ within the control volume at time t, is modelled by the solution of differential equation $\frac{dm}{dt}=q_{\text{in}}\times c_{\text{in}}-q_{\text{out}}\times c_{\text{out}}$.

To solve this equation, the mass concentration of the solute in the outflow c_{out} must be expressed in terms of the dependent variable $m(t)$ and the control volume $V(t)$ as $c_{\text{out}}=\frac{m(t)}{V(t)}$.

However, the net rate of change of the volume of the solution within the control volume is modelled by the simple differential equation $\frac{dV}{dt}=q_{\text{in}}-q_{\text{out}}$, which gives the solution $V(t)=V(0)+\left(q_{\text{in}}-q_{\text{out}}\right)t$.

The differential equation for the time rate of change of the conserved substance within the control volume can now be written as $\frac{dm}{dt}=q_{\text{in}}\times c_{\text{in}}-q_{\text{out}}\times\frac{m(t)}{V(0)+\left(q_{\text{in}}-q_{\text{out}}\right)t}$.

Modelling the mass of a conserved quantity

Example 32

A holding tank at a desalination plant contains 200 cubic metres of brine solution (salt in water), 20 tonnes of which is salt. At time $t=0$ minutes, seawater starts to flow into this at a rate of 10 cubic metres per minute. This seawater has a salt concentration of 35 kilograms per cubic metre. The 'well-mixed' solution (brine mixed with seawater) also flows out of the holding tank at the same rate of 10 cubic metres per minute.

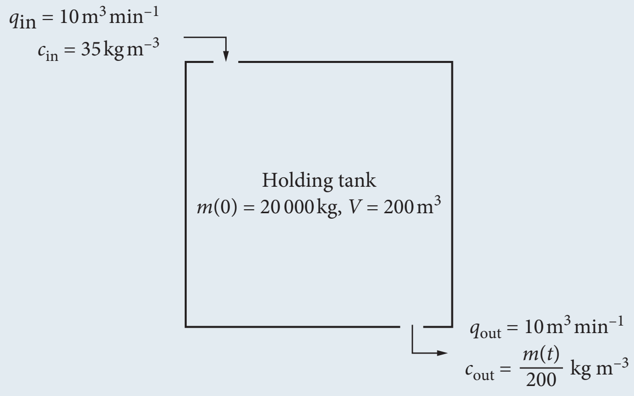
(a) Construct a differential equation to model the mass m kg of salt in the tank after t minutes.

(b) State the amount of salt within the holding tank $m(t)$ kg after t minutes.

(c) As the water continues to flow through the tank, over time the concentration of salt within the tank approaches a 'long-term' value. What is this value, in units of kg m^{-3}?

(d) Sketch the time dependence of the salt concentration of the 'well-mixed' outflow.

Solution



(a) The volume of the solution is fixed at 200 m^3, because $q_{\text{in}}=q_{\text{out}}=10\text{ m}^3\text{ min}^{-1}$.

Concentration of the inflow $c_{\text{in}}=35\text{ kg m}^{-3}$.

Concentration of the outflow $c_{\text{out}}=\frac{m(t)}{200}\text{ kg m}^{-3}$, where $m(t)$ is the kilograms of salt within the tank after t minutes.

The initial mass of salt is $m(0)=20\,000$ kg.

$$\text{Net rate of change of salt}\left(\frac{\text{kg}}{\text{min}}\right)=\text{rate of inflow}\left(\frac{\text{kg}}{\text{min}}\right)-\text{rate of outflow}\left(\frac{\text{kg}}{\text{min}}\right)$$

$$\begin{aligned}\frac{dm}{dt}&=350-\frac{m}{20}\\&=\frac{1}{20}(7\,000-m)\ \text{ given that } m(0)=20\,000\text{ kg}\end{aligned}$$

(b) The model is an example of the general inhibited decay model:

$$\frac{dy}{dt}=r(a-y),\ y(0)=y_0\ \therefore\ y=a+(y_0-a)e^{-rt}$$

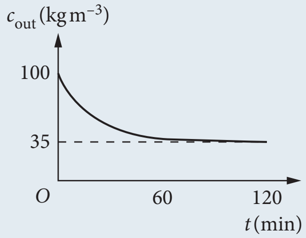
and thus has the solution $m(t)=7000+13\,000e^{\frac{-t}{20}}$

(c) Any decaying exponentials will vanish over time:

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} 1000\left(7+13e^{\frac{-t}{20}}\right) = 7000\text{ kg}$$

(d) The concentration c_{out} of the outflow is the mass divided by the volume of the tank: $c_{\text{out}} = \dfrac{m}{200}\text{ kg m}^{-3}$.

The mass exhibits an inhibited decay towards its equilibrium value of 7000 kg. This corresponds to a mass concentration of 35 kg m^{-3}.



EXERCISE 14.7 MODELLING WITH FIRST-ORDER DIFFERENTIAL EQUATIONS

1 Market research in a large city indicates that the maximum sales of a soon-to-be-released mobile device, is 10 truckloads per month (1 truckload = 10 000 devices).

Past experience with models iThingie1 through to iThingie6 indicates that the rate of growth in the truckloads of sales $\dfrac{ds}{dt}$, t months after the release of an iThingie, is directly proportional to the difference between the current sales and the maximum monthly sales.

(a) Find an equation for the rate of growth $\dfrac{ds}{dt}$ in the sales, s, as a function of the time, t, in months after the new product is first released onto the market. Express your answer in terms of the constant of proportionality, r.

(b) Find the solution curve of your model. Express your answer in terms of the constant of proportionality, r.

(c) If two truckloads are sold after one month, find the predicted number of truckloads per month after three months. (Express your answer correct to the nearest truckload.)

2 A simple model for the spread of a contagious illness assumes that the rate at which the illness spreads $\dfrac{dI}{dt}$ varies jointly with the product of the number of ill people I and the number of people still susceptible to the illness S. This means that $\dfrac{dI}{dt} = rIS$, $r > 0$.

Assume that one infected person is introduced into a fixed population of size P.
Then $P+1 = I+S$, $\therefore\ S = P+1-I$. Therefore, $\dfrac{dI}{dt} = rI\left(P+1-I\right)$, $I(0) = 1$ and $r > 0$.

(a) Show that $\dfrac{1}{I\left(P+1-I\right)} = \dfrac{1}{(1+P)}\left[\dfrac{1}{\left(1+P-I\right)} + \dfrac{1}{I}\right]$.

(b) Find I as a function of time.

3 A pond initially contains 200 000 litres of unpolluted water. A stream begins to flow into the pond at rate of 10 000 litres per day. The stream is polluted with a concentration of 2 grams of pollutant per litre. The pond also has an outlet that spills 10 000 litres of well-mixed water per day.

(a) State the initial value problem that models the mass of pollutant $m(t)$ grams in the pond, t days after the polluted stream first begins to flow into the pond.

(b) Hence find a differential equation that models the concentration of pollutant $c(t) = \dfrac{m(t)}{200\,000}$ grams per litre in the pond, t days after the polluted stream first begins to flow into the pond.

(c) Solve the model from part **(a)**.

(d) What is the concentration of pollutant in the pond after 10 days?

4 A tank initially contains 1000 litres of salt solution of concentration 0.01 kg/L. A solution of the same salt, but concentration 0.04 kg L^{-1}, flows into the tank at a rate of 10 litres per minute. The mixture in the tank is kept uniform by stirring and the mixture flows out at a rate of 5 litres per minute.

Let Q kg be the quantity of salt in the tank after t minutes. Set up (but do not solve) the differential equation for Q in terms of t, and specify the initial conditions.

5 Carbon monoxide (chemical symbol CO) is toxic to humans. Two hours of exposure to air with a volume concentration of CO at 0.02% will cause headaches and confusion.

During World War I, some generals commanded their soldiers from inside a bombproof bunker with an internal volume of 80 m^3. Troops resting near the air intake to the bunker would often smoke cigarettes. Unfortunately, the air intake to the bunker sucked the carbon-monoxide-filled smoke from the cigarettes back into the bunker.

Assume that smoky air is sucked into the bunker at a rate of $2\,\text{m}^3\,\text{min}^{-1}$, and that 0.03% of this air (by volume) is carbon monoxide. Ventilation fans keep the air well mixed inside the bunker, and the well-mixed air is extracted from the bunker at the same rate of $2\,\text{m}^3\,\text{min}^{-1}$. It can be shown that $\frac{dv}{dt} = 0.025(0.0003 - v)$, $v(0) = 0$.

(a) Solve this differential equation to find $v(t)$.

(b) Hence find the time for the volume fraction of carbon monoxide to reach 0.02% by volume inside the bunker. Express your answer in minutes, correct to the nearest minute.

6 A lottery winner puts \$5 000 000 in winnings into a fund that has a 5% annual rate of return, paid continuously throughout the year. Each year the winner spends \$300 000, withdrawn from the account at a continuous rate over the course of the year.

(a) Show that the differential equation to model the fund balance $x(t)$ after t years is $\frac{dx}{dt} = 0.05(x - 600\,000)$ and state the value of $x(0)$.

(b) Solve the differential equation in part **(a)**.

(c) Hence determine the balance after 20 years. Express your answer correct to the nearest 5 cents.

7 Which of the following is not a model for exponential growth y?

A $y = \frac{10}{e^{-t}},\ t > 0$

B $y = 10(0.5)^t,\ t > 0$

C $\log_e \frac{1}{y} = \log_e 10 - 2t,\ t > 0$

D $\log_e(y) = \log_e(10) + t,\ t > 0$

8 A can of soft drink at a temperature of 24°C is placed in a freezer. The temperature inside the freezer is maintained at −4°C. When the can has been in the freezer for t minutes, the temperature of the can is T°C. The rate of decrease in the temperature of the can is proportional to the excess of its temperature over the temperature inside the freezer.

If k is a positive constant, the temperature of the can is modelled by the solution of:

A $\frac{dT}{dt} = -k(T - 24);\quad t = 0, T = -4$

B $\frac{dT}{dt} = -k(T - 24);\quad t = 0, T = -4$

C $\frac{dT}{dt} = -k(T + 4);\quad t = 0, T = 24$

D $\frac{dT}{dt} = -k(T - 4);\quad t = 0, T = 24$

9 The rate at which a rumour spreads throughout a population of 1000 students is proportional to the product of the number N of students who know the rumour and the number of students who haven't yet heard the rumour after t hours. If two students decide to start a rumour, the model that best decribes the spread of the rumour t hours later is:

A $\frac{dN}{dt} = k\frac{(1000 - N)}{1000},\quad N(0) = 0$

B $\frac{dN}{dt} = k(N - 2)(1000 - N),\quad N(0) = 2$

C $\frac{dN}{dt} = kN(1000 - N),\quad N(0) = 0$

D $\frac{dN}{dt} = kN(1000 - N),\quad N(0) = 2$

10 A chemical dissolves in a pool at a rate equal to 10% of the amount of undissolved chemical. Initially the amount of undissolved chemical is 5 kg and after t hours x kilograms has dissolved. The differential equation that models this process is:

A $\frac{dx}{dt} = \frac{x}{10}$

B $\frac{dx}{dt} = \frac{5 - x}{10}$

C $\frac{dx}{dt} = \frac{x - 5}{10}$

D $\frac{dx}{dt} = 5 - \frac{x}{10}$

11 The rate of increase in the number of bacteria in a laboratory is directly proportional to the number present. If the number of bacteria triples every 2 hours, after how many hours will the number of bacteria be quadruple its initial value?

A $2\log_e \frac{4}{3}$ **B** $\frac{2\log_e 4}{\log_e 3}$ **C** $\left(\frac{\log_e 3}{2}\right)^2$ **D** $\frac{\log_e 3}{\log_e 2}$

12 In a simple model for the body mass, *m*, of an adult, the rate of change of body mass in kg day^{-1} varies directly with the difference between the total energy (food) intake, *C*, per day and the total energy expended per day. The energy expenditure per day depends upon the metabolic rate of the individual, but 165 kJ kg^{-1} of body mass is a realistic average value.

Assuming *t* is measured in days, the differential equation modelling the rate of change of body mass in kg day^{-1} could be:

A $\frac{dm}{dt} = k(165 - C), \quad k > 0$ **B** $\frac{dC}{dt} = k(165m - C), \quad k > 0$

C $\frac{dm}{dt} = k(C - 165m), \quad k > 0$ **D** $\frac{dC}{dt} = k(C - 165m), \quad k > 0$

13 A population $P(t)$ of an animal satisfies $\frac{dP}{dt} = \frac{1}{10}P\left(2 - \frac{P}{500}\right)$, $P(0) = 100$ with *t* measured in years. What is $P(t)$ as $t \to \infty$? You are given that $\frac{1}{P(1000-P)} = \frac{1}{1000}\left(\frac{1}{P} - \frac{1}{1000-P}\right)$.

A 25 **B** 250 **C** 500 **D** 1000

14 The growth rate of a tree varies jointly with the product of the current height *h* and the difference between the current height and the maximum height *H*. The differential equation modelling this growth could be:

A $\frac{dH}{dt} = r(h - H), \quad r > 0$ **B** $\frac{dH}{dt} = rh(H - h), \quad r > 0$

C $\frac{dh}{dt} = rh(h - H), \quad r > 0$ **D** $\frac{dh}{dt} = rh(H - h), \quad r > 0$

15 At any time $t \geq 0$ (in days), the rate of growth in the number of bacteria in a laboratory is directly proportional to the number *N* currently present. The initial population of bacteria is 1000.

(a) Assuming this growth rate continues indefinitely, write a differential equation to model the number of bacteria present in the dish after $t \geq 0$ days.

The initial population of 1000 bacteria triples during the first 2 days.

(b) Hence, show that $N(t) = a \times 3^{\frac{t}{b}}$ for a suitable choice of the positive integers *a* and *b*.
(c) By what factor will the population have increased in the first 4 days?
(d) How much time will it take for the population to grow to 10 times its initial value? Express your answer correct to the nearest hour.

Challenging

16 Almost all carbon in the world is carbon-12, which is the most common stable 'isotope' (nuclear form) of carbon. In the late 1940s the American scientist Willard Libby studied carbon-14, which is not stable: it radioactively decays according to the reaction ${}^{14}\text{C} \to {}^{14}\text{N} + \text{e}^- + \bar{\nu}_\text{e}$, in which a neutron spontaneously transforms into a proton (thus changing the atom from carbon C to nitrogen N) as it emits an electron and an antineutrino. In the upper atmosphere, carbon-12 sometimes transforms back into carbon-14 due to interactions with cosmic rays, so the proportion of both isotopes in the atmosphere stays relatively constant.

But whenever carbon is absorbed by plants to become part of living organisms in the world the carbon-12 is mostly shielded from transforming into carbon-14. This means that when an organism dies, its concentration of carbon-12 $\left(\left[{}^{12}\text{C}\right]\right)$ remains relatively constant, but its concentration of carbon-14 $\left(\left[{}^{14}\text{C}\right]\right)$ radioactively decays at the rate $\frac{d\left[{}^{14}\text{C}\right]}{dt} = -r\left[{}^{14}\text{C}\right]$, $r = 1.2097 \times 10^{-4}$ years^{-1}.

(a) Find the half-life $\left(t_{\frac{1}{2}}\right)$ of carbon-14, correct to the nearest year.

The ratio of carbon-12 to carbon-14 remains relatively constant in living organisms, roughly $R = \frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]} \approx 1.3 \times 10^{-12}$, but this ratio changes after the organism dies (because it stops absorbing new carbon-14 atoms from the atmosphere, while any carbon-14 present is still decaying). Consequently, you can determine the length of time since an organism's death by measuring how much this ratio has changed.

(b) Half the original carbon-14 has radioactively decayed. How many years ago did the tree die?

(c) Find a differential equation for the ratio $R = \frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]}$. (Hint: $\left[{}^{-12}\text{C}\right]$ can be considered a constant.)

(d) Hence find a formula for the ratio R after t years in the form $R(t) = a\left(\frac{1}{2}\right)^{\frac{t}{n}}$ for a suitable choice of real number a and integer n.

(e) The skeleton of an extinct mega-marsupial is found to have a carbon ratio $\frac{\left[{}^{14}\text{C}\right]}{\left[{}^{12}\text{C}\right]} = 0.9 \times 10^{-4}$. How many years ago, to the nearest year, did the animal die?

17 Bubonic plague (known as the Black Death) ravaged Europe most severely between the years 1347 and 1351. It is estimated that the plague killed more than one in three people living in Europe at that time. The disease was survivable only in about 5% of cases.

Epidemiologists commonly use S to represent the fraction of a population that have survived an epidemic disease, t days after its arrival. The change in this fraction over time can be modelled by the differential equation: $\frac{dS}{dt} = -r(S - I)$, $S(0) = 1$, where I is the fraction of the population that ultimately recover (and hence survive).

(a) Find a formula for the survivability fraction S as a function of the time t days.

(b) Show that $\frac{d^2S}{dt^2} = r^2(S - I)$

After 1 month, the survivability fraction S approaches a steady state value of 0.05.

(c) Find I.

(d) Find the value of r if after 14 days, only 6% of the population has survived.

(e) Find the time when the death rate reaches its maximum value and state this death rate.

(f) Plot the survivability fraction over the first 2 weeks.

18 Two types of bacteria, type A and type B, coexist in a biological system. Assume that each population grows exponentially. The proportion of the total bacteria population belonging to type A is given by $p(t) = \frac{A(t)}{A(t) + B(t)}$.

(a) Express the growth rate $\frac{dp}{dt}$ in terms of $A(t)$, $B(t)$, $A'(t)$ and $B'(t)$.

(b) Given that $A'(t) = r_A A(t)$, $r_A > 0$ and $B'(t) = r_B B(t)$, $r_B > 0$, express the growth rate $\frac{dp}{dt}$ in terms r_A, r_B, A and B.

(c) Hence write the growth rate $\frac{dp}{dt}$ in terms r_A, r_B and p.

(d) If $p(0) = \frac{1}{10}$ and $r_A - r_B = \frac{1}{100}$ hour^{-1}, and given that $\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}$, find $p(10)$.

19 Toxins that enter an ecosystem are generally observed to be more concentrated per unit of biomass as you move up the food chain. This phenomenon is known as bioaccumulation.

Following one growth rate model, the length L (m) of a species of tuna at age t (years) is given by $\frac{dL}{dt} = \frac{1}{5}(3 - L)$, $L(0) = 0$, $0 \le t \le 20$.

(a) Find $L(t)$.

The weight W (kg) of a tuna is related to its length L by the equation $W(t) = 16L^3$. Let $H(t)$ (mg) be the accumulated mass of mercury (in milligrams) in a tuna after t years. The rate at which mercury is added to the tissue of the tuna is $\frac{dH}{dt} = \frac{1}{100}W$, $H(0) = 0$.

(b) Find $H(t)$.

(c) Find a formula for the concentration C of mercury in units of milligrams of mercury per kilogram of tuna, for a tuna of age t years.

(d) Assuming the tuna has a lifespan of 20 years, plot the mercury concentration in the tuna over its lifetime, $0 < t \le 20$.

20 The British actuary and mathematician Benjamin Gompertz (1779–1865) proposed the following growth rate model: $\frac{dW}{dt} = rW\log_e\left(\frac{C}{W}\right)$ [1]

Numerous experimental studies have demonstrated that the growth rate $\frac{dW}{dt}$ of tumours is modelled by

$\frac{dW}{dt} = \frac{1}{20}W\left(10 - \log_e W\right)$, $W\left(0\right) = e$ [2]

where W is the weight of the tumour (in milligrams) and t is the time in days.

(a) Given that W grows according to the Gompertz growth rate model [1], find the Gompertz parameters r and C from [2].

(b) Verify that $W = e^{10 - 9e^{\frac{-t}{20}}}$ is the solution of the differential equation [2].

(c) Find the equilibrium weight of the tumour, i.e. the weight of the tumour when it stops growing.

(d) Find the maximum growth rate for this tumour.

Treatment of many tumours is most effective when the tumour is growing at its fastest rate.

(e) After how long is the tumour growing at the fastest rate?

(f) Sketch the graph of W for the first 3 months, showing any equilibrium solutions and the point of maximum growth rate.

21 Oil is pumped from a Bass Strait oil well at a rate proportional to the volume V of oil (in units of 'barrels') remaining in the well after t years. Initially, the well had 1 000 000 barrels of oil, but 5 years later, only 600 000 barrels of oil remain. It will not be profitable to continue pumping oil when fewer than 1000 barrels of oil remain.

(a) Show that the volume of oil V remaining after t years is given by $V(t) = a\left(\frac{b}{c}\right)^{\frac{t}{d}}$ for a suitable choice of positive integers a, b, c and d.

(b) At what rate is the remaining number of barrels of oil decreasing after 5 years?

(c) For how many years will the oil well remain profitable? Express your answer to the nearest month.

22 A nature conservation group releases 21 Tasmanian devils onto a remote island off the coast of Tasmania. The group believes the island can support at most 588 Tasmanian devils. The growth rate of the Tasmanian devil population p is $\frac{dp}{dt} = rp\left(1 - \frac{p}{588}\right)$, $p\left(0\right) = 21$, with $r > 0$ and t measured in years.

(a) Show that $\frac{1}{p\left(1 - \frac{p}{588}\right)} = \frac{1}{p} - \frac{1}{p - 588}$.

(b) State the model for the Tasmanian devil population p in terms of r and time t.

Three years after the beginning of the breeding program, the population is 294.

(c) Find r.

(d) Use the model from part **(c)** to estimate the devil population after 6 years.

Six years after the Tasmanian devils are first taken to the island, the nature conservation group decides it is time to repopulate the mainland with the island's devils. A decision is made to take 140 devils from the island each year.

(e) Find a differential equation $\frac{dP}{dt} = f(P)$ for a suitable choice of the function $f(P)$ that models the modified growth rate in the island population $P(t)$, t years after the devils were first taken from the island back to the mainland. Assume that the value of r and the carrying capacity of devils on the island both remain unchanged.

(f) If $\frac{dP}{dt} = -K(P-a)(P-b)$, $P(0) = P_0$, find $P(t)$, given that $\frac{1}{(P-a)(P-b)} = \frac{1}{b-a}\left(\frac{1}{P-b} - \frac{1}{P-a}\right)$.

More challenging question parts:

(g) Express the function $f(P)$ found in part **(e)** in the form $f(P) = -\frac{\log_e 3}{588}(P-a)(P-b)$, $a < b$, for a suitable choice of real numbers a and b.

(h) Hence find a solution curve for the island population $P(t)$, t years after the decision to repopulate the mainland.

(i) Plot the Tasmanian devil population over the first 14 years of the breeding program. Hence, comment on the effectiveness of the breeding program.

CHAPTER REVIEW 14

1 A camera at ground level is 400 metres away from a hot air balloon just prior to the balloon lifting off. The balloon lifts off and the camera records the balloon rising into the sky at a constant rate of 10 metres per second.

(a) If θ is the angle of elevation of the balloon, express the height h of the balloon in terms of this angle.

(b) How fast is the angle of elevation θ radians changing when the balloon is 300 m above the ground?

2 Verify that $y = e^{-x}\cos x$ is a solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

3 The gradient of the tangent to a curve at any point (x, y) is $\frac{x}{x+1}$, $x > -1$. If the curve passes through the point $(1, 1)$, find the equation of the curve.

4 **(a)** If $f''(x) = \cos 2x - 3\sin x$ and $f'(0) = 3$, find $f'(x)$.

(b) Hence if $f\left(\frac{\pi}{2}\right) = 3$, find $f(x)$.

5 Solve the differential equation $\frac{dy}{dx} = 1 - 2y$, given that $y = -1$ where $x = 0$.

6 **(a)** Show that $\frac{4}{y^2-4} = \frac{1}{y-2} - \frac{1}{y+2}$.

(b) Find y as a function of x if y satisfies the differential equation $\frac{dy}{dx} = y^2 - 4$ and $y = 0$ where $x = 0$.

7 A species of tuna is declining so that T, the number of tuna at a time t years from now, satisfies the differential equation $\frac{dT}{dt} = -0.1T$.

(a) Write the general solution to this differential equation, where $T(0) = A > 0$ is the initial population.

(b) Find the time it will take for the numbers to fall to one-quarter of their present value.

8 Consider the initial value problem $\frac{dy}{dx} = 2x\left(1 + y^2\right)$, $y(0) = 1$. Find the exact solution to the differential equation.

9 If $\sin x = e^y$, $0 < x < \pi$, what is $\frac{dy}{dx}$ in terms of x?

A $-\cot x$ **B** $\tan x$ **C** $-\tan x$ **D** $\cot x$

10 Which one of the following differential equations is *not* satisfied by $x = e^{-3t}$?

A $\left(\frac{dx}{dt}\right)^2 - 9x^2 = 0$ **B** $\frac{d^2x}{dt^2} - 9x = 0$ **C** $\frac{d^2x}{dt^2} + 9x = 0$ **D** $\frac{dx}{dt} + 3x = 0$

11 If $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$, then the possible values of k are:

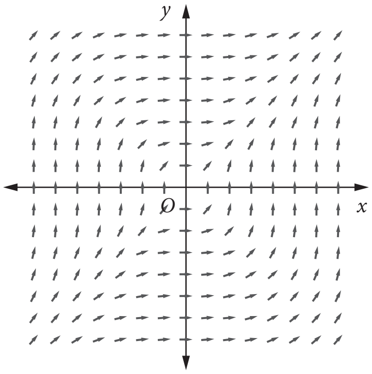
A 1 and −6 **B** −1 and 6 **C** −2 and 3 **D** 2 and −3

12 If $\frac{dy}{dx} = 2 - 3\sqrt{x}$ and $y = 3$ where $x = 1$, then y is given by the rule:

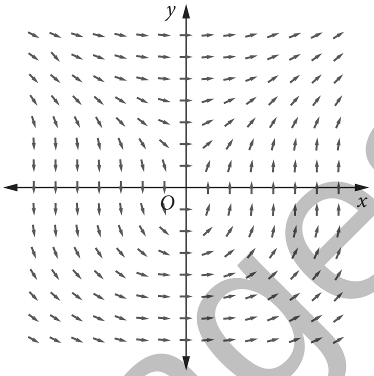
A $1 + 3x(1 - \sqrt{x})$ **B** $3 + 2x(1 - \sqrt{x})$ **C** $3 + 2x + 2x\sqrt{x}$ **D** $6\sqrt{3} - 5 + 2x - 2x\sqrt{x}$

13 What is the slope field of $y' = \frac{x^2}{y^2}$?

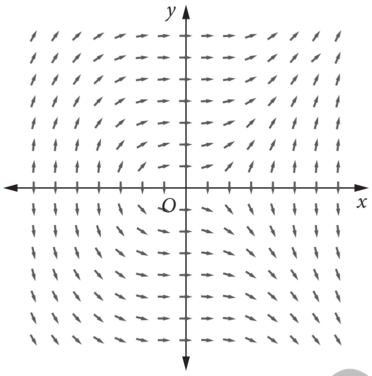
A



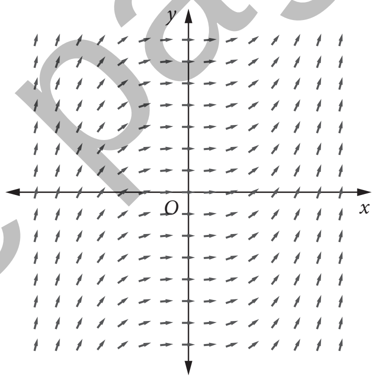
B



C



D



14 If $\frac{dy}{dx} = -3y$, and $y = 1$ where $x = 2$, then which of the following is true?

A $y = 2e^{3-3x}$ **B** $y = e^{6-3x}$ **C** $y = 2e^{3x-3}$ **D** $y = e^{3x-6}$

15 The general solution of $\frac{dy}{dx} = -(2 + y)$ is:

A $y = e^{-x} + A$, A is a real number **B** $y = Ae^{-x} + 2$, A is a real number

C $y = Ae^{x} + 2$, A is a real number **D** $y = Ae^{-x} - 2$, A is a real number

16 Consider the differential equation $\frac{dy}{dx} = x - 2y$, for which the solution is $g(x)$. Which of the following statements about the particular solution that contains the point $(0, -1)$ is true at $x = 0$?

A the graph is increasing and concave up **B** the graph is increasing and concave down

C the graph is decreasing and concave up **D** the graph is decreasing and concave down

Questions 17 and 18 refer to the following information.

Consider the differential equation $\frac{dy}{dx} = y\sin x$, for which the solution is $y = f(x)$. Let $f(0) = 1$.

17 Which of the following statements about the graph of $f(x)$ are true?

(i) The slope of $f(x)$ at the point $\left(\frac{\pi}{2}, 1\right)$ is 1.

(ii) $f(x)$ has a horizontal tangent where $x = 0$.

(iii) $f(x)$ has a vertical tangent where $y = 0$.

A **(i)** only **B** **(ii)** only **C** **(i)** and **(ii)** only **D** **(ii)** and **(iii)** only

18 The particular solution is:

A $y = e^{1-\cos x}$ **B** $y = e^{\cos x - 1}$ **C** $y = e^{-\sin x}$ **D** $y = e^{\sin x}$

19 When added to water, 5 grams of a substance dissolves at a rate equal to 10% of the amount of undissolved chemical per hour. If x is the number of grams of *undissolved* chemical after t hours, then x satisfies the differential equation:

A $\frac{dx}{dt} = -\frac{1}{10}x$ **B** $\frac{dx}{dt} = -\frac{1}{5}x$ **C** $\frac{dx}{dt} = \frac{1}{5}(10-x)$ **D** $\frac{dx}{dt} = \frac{1}{10}(5-x)$

20 The population $P(t)$ of a certain species satisfies the differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{10\,000}\right)$ with an initial population $P(0) = 4000$, where t is the time in years. What is the population as t approaches infinity?

Note that $\frac{1}{P} + \frac{1}{20\,000 - P} = \frac{20\,000}{P(20\,000 - P)}$.

A 4000 **B** 5000 **C** 10 000 **D** 20 000

21 A quantity of sugar is dissolved in a tank containing 100 litres of pure water. At time $t = 0$ minutes, pure water is poured into the tank at a rate of 4 litres per minute. The tank is kept well stirred at all times. At the same time, the sugar solution is drained from a tap at the bottom of the tank at a rate of 6 litres per minute. A differential equation for the mass m grams of sugar in the tank is:

A $\frac{dm}{dt} = -6m$ **B** $\frac{dm}{dt} = 4 - \frac{3m}{50}$ **C** $\frac{dm}{dt} = -\frac{3m}{50-t}$ **D** $\frac{dm}{dt} = 4 - \frac{3m}{50-t}$

22 A cup of hot coffee at a temperature T°C loses heat in a cooler environment. It cools according to the law $\frac{dT}{dt} = -k(T - T_0)$, where t is the time elapsed in minutes, T_0 is the temperature of the environment in degrees Celsius and k is a constant.

(a) At recess, Mr Masters makes his cup of coffee with water at 100°C. The temperature in the staff room is 25°C when he places his coffee on his desk. Three minutes later his coffee is just the temperature he likes it, 75°C. Find the value of k.

(b) Before Mr Masters gets a chance to drink his coffee, he leaves the staff room to help a student with a maths problem about exponential growth and decay. What is the temperature of the coffee when he returns, 5 minutes later?

23 Cream taken out of a refrigerator has a temperature of 3°C. It is placed on the table in a room of constant temperature 24°C. After t minutes the temperature, T°C, of the cream is given by $T = A - Be^{-0.04t}$, where A and B are positive constants.

How much time does it take for the cream to reach a temperature of 8°C?

24 According to Fourier's law of heat conduction, the rate of heat transfer $\frac{dQ}{dt}$ through an ice sheet in Antarctica is given by the differential equation $\frac{dQ}{dt} = \frac{k(T_w - T_a)}{h}$, where k is the thermal conductivity of the ice, h is the thickness of the ice sheet, and T_w and T_a are the temperatures at the ice/water boundary and the ice–air boundary respectively.

As the water loses Q joules of heat through the ice sheet, the rate of increase in ice thickness h is given by $\frac{dh}{dQ} = \frac{1}{L\rho}$, where L is the latent heat of sea water (in other words, the amount of heat loss required to freeze 1 kilogram of it) and ρ is the density of the ice.

(a) Find the rate of increase of the ice sheet thickness $\frac{dh}{dt}$.

(b) If $h(0) = h_0$, find $h(t)$, assuming that $\frac{k(T_w - T_a)}{L\rho}$ is a positive constant.

25 The quantity q sold of a new product is a function of the selling price p. The revenue $R(p) = p\,q(p)$ from selling q units at price p is also a function of the selling price. It can be shown that $\frac{dR}{dp} = q\left(1 + \frac{p}{q}\frac{dq}{dp}\right)$.
Economists call $E = \frac{p}{q}\frac{dq}{dp}$ the *price elasticity* of demand q with respect to price p. This price elasticity of demand measures the relative change of the quantity demanded in response to given relative change in price.

(a) Show that revenue is a maximum where $E = -1$.

The inverse demand curve of a product measures the selling price $p(q)$ in terms of the available supply q. The law of demand states that, 'when the price of a good rises, and everything else remains the same, the quantity of the good demanded will fall'. In other words, selling price varies inversely with the demanded supply.
Suppose that a business has the exclusive right to import a new product into a large city. Market research indicates that the selling price \$$p$ for a new product is modelled by $p = \sqrt{25 - q^2}$, where the supply q is measured in units of 1000.

(b) State the domain of the model.

(c) Find the rate of change of the demanded quantity q with respect to the selling price p. (Express your answer in terms of q.)

(d) Hence find the selling price that will maximise revenue.

26 The relationship between air temperature T K and wind speed V m s^{-1} is modelled by the differential equation $\frac{dT}{dV} = -\frac{V}{C_p}$, where C_p is a constant known as the specific heat of the air mass.

(a) Find the general solution for the wind speed $V(T)$ as a function of temperature.

Near a particular tropical cyclone the wind speed is zero and the temperature is 308 K.

(b) Find the particular solution for the wind speed $V(T)$ as a function of temperature.

(c) If $C_p = 1004.6$, find the wind speed near the centre of a tropical cyclone if the temperature there is 307 K.

27 A mathematical model for the relationship between x the number of predators (in hundreds) and y the number of prey (in thousands), in a particular environment at time, t, years, gives the following pair of differential equations: $\frac{dy}{dt} = -\frac{1}{400}(x-2)$, $\frac{dx}{dt} = \frac{1}{100}(y-1)$

(a) Use the chain rule to obtain a differential equation involving $\frac{dy}{dx}$, x and y only.

(b) At some point in time there are 100 predators ($x = 1$) and 3000 prey ($y = 3$). Hence show that $a^2(x-h)^2 + b^2(y-k)^2 = 1$ for real numbers a, b, h and k.

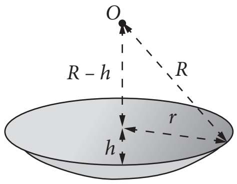
(c) The graph of the solution in part **(b)** is an ellipse with a centre at (h, k). Sketch your solution from part **(b)**.

(d) From your previous answers, find the maximum and minimum number of predators the environment can support.

28 This question uses data from the table below.

$V(h) = \frac{1}{3}\pi h^2(3R-h)$	Volume of spherical cap (m^3)
$A_1(h) = \pi h(2R-h)$	Free surface area of a spherical cap (m^2)
$A_2(h) = 2\pi hR$	Curved surface area of a spherical cap (m^2)
ε	Evaporation coefficient for lake surface (m^3 m^{-2} day^{-1})
σ	Seepage coefficient for lake bottom (m^3 m^{-2} day^{-1})
Q_m	Net rate of inflow from rivers (m^3 day^{-1})

The Lake Eyre basin is the lowest point in Australia and usually dry. However, the southern lake of the Lake Eyre complex fills to a depth of about 3 metres via inflow from flooding rivers, a few times each century.



This southern lake can be modelled as the cap of a sphere of radius R (m), with a centre at O, as shown. Assume the inflow from the flooding rivers has stopped so that $Q_m \approx 0$ (m^3 day^{-1}).

A possible balance for the volume of water in the southern lake of the Lake Eyre complex during a flood year is $\dfrac{dV}{dt} = Q_m - \varepsilon A_1(h) - \sigma A_2(h)$.

(a) Use the information from the table above to express $\dfrac{dV}{dt}$ as an explicit function of h.

(b) Show that $\dfrac{dV}{dh} = \pi h(2R - h)$ and hence find $\dfrac{dh}{dt}$ in simplest form.

The depth of the water in the lake can now be modelled immediately after an inflow of 1 gigalitre (1 000 000 m^3), which fills the lake to a depth of 3 m above its centre so that $h_0 = 3$ m. (Assume no further inflows or precipitation.)

(c) Find the radius R of the hemispherical cap.

29 The coroner arrives at Frogmorton Manor on a cold winter evening. Lady Frogmorton has just been found dead in her climatically controlled greenhouse by the butler. The temperature of the body is taken, and is recorded as 29°C at precisely 7:20 pm. The coroner checks the programmable thermostat and determines the greenhouse has been kept at a constant 20°C for the past 12 months. At precisely 8:20 pm, the coroner takes the temperature of the body once more, and it is recorded as 27.4°C.

Smiggins is Lady Frogmorton's chauffer. Under questioning by the police, he claims that he delivered Lady Frogmorton to the manor at around 6 pm that evening, after a day at the Chelsea Flower Show.

(a) Assuming a normal body temperature of 37°C, propose a differential equation to model the temperature $T(t)$ of the deceased at a time t hours since death.

(b) Solve the initial value problem in part **(a)**. Recall that the coroner arrived on the scene at precisely 7:20 pm, which is τ hours since death.

(c) Show that at τ hours after death $e^{r\tau} = \dfrac{17}{9}$.

(d) Find an expression, similar to your answer in part **(c)**, for the temperature at 8:20 pm.

(e) Hence find r.

(f) The coroner informs the police that the chauffer's story may need further investigation. Explain why this is so.

Challenging

30 Bob's credit card bill B is initially \$15 000 and he pays 18% interest on this debt per year, compounded continuously. He decides to pay it off by transferring money from his savings account continuously at the rate of \$300 per month.

(a) Find and solve a differential equation to model the credit card balance B after t years.

(b) How much time will it take to pay off the credit card bill (to the nearest day)?

(c) What is the sum total of Bob's repayments?

Assume Bob has \$40 000 in a savings account that accumulates interest at an annual rate of 6%, also compounded continuously.

(d) Find and solve a differential equation to model the balance S of Bob's savings account.

(e) How much money will Bob have in his savings account when the debt is finally paid off (assuming no other transactions)?

31 An abandoned open-cut mine just outside a large city has been purchased as a landfill for solid waste by a city council. When purchased, the open-cut mine had a volume of 1 million cubic metres. At the beginning of 2015, the landfill already had 100 000 cubic metres of solid waste. The volume of solid waste W in the landfill (measured in units of 100 000 cubic metres) t years after the beginning of 2015 is modelled by the solution of the differential equation $\frac{dW}{dt}=\frac{1}{10}(10-W),\ W(0)=1$.

(a) Find the volume of solid waste in the landfill t years after 2015.

(b) Hence determine the volume of solid waste in the landfill at the beginning of 2035. (Express your answer in cubic metres, correct to the nearest cubic metre.)

32 The population $P(t)$ of penguins on an island in the Southern Ocean t years after the beginning of 2015 grows at a rate directly proportional to $1000 - P(t)$, where the constant of proportionality is k.

(a) If the population at the beginning of 2015 is 200, express the penguin population t years after the beginning of 2015 in terms of t and k.

(b) If the population after 2 years is 300, find k.

(c) Hence determine the long-term population of penguins on the island.

33 While on an unauthorised trip to the local fast food restaurant during their study period, a pair of Year 12 students are convinced that they have just seen the Prime Minister buying a hamburger. On returning to the school, their amazing discovery spreads throughout the school community at the rate $\frac{dp}{dt}=\frac{1}{10}p(1-p)$, where p is the proportion of the school community that has already heard the rumour, t minutes after their return to school.

(a) What proportion of the school community has heard the rumour when it is spreading most rapidly?
By the beginning of the afternoon period, 20% of the school community had already heard the rumour.

(b) Find $p(t)$, at time t minutes since the beginning of the afternoon period, given $\frac{1}{p(1-p)}=\frac{1}{p}-\frac{1}{p-1}$.

(c) At what time (correct to the nearest minute) is the rumour spreading most rapidly?